TWISTED COMPONENT SUMS OF VECTOR-VALUED MODULAR FORMS

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Abstract. We construct isomorphisms between spaces of vector-valued modular forms for the dual Weil representation and certain spaces of scalar-valued modular forms in the case that the underlying finite quadratic module A has order p or $2p$, where p is an odd prime. The isomorphisms are given by twisted sums of the components of vector-valued modular forms. Our results generalize work of Bruinier and Bundschuh to the case that the components F_γ of the vector-valued modular form are antisymmetric in the sense that $F_\gamma = -F_{-\gamma}$ for all $\gamma \in A$. As an application, we compute restrictions of Doi-Naganuma lifts of odd weight to components of Hirzebruch-Zagier curves.

1. Introduction

In the study of theta lifts (such as Maass lifts and Borcherds products) it is convenient to work with vector-valued modular forms for the dual Weil representation ρ^* associated to a finite quadratic module (A, Q). Therefore, it is useful to understand the precise relationship between vector-valued modular forms for ρ^* and scalar-valued modular forms for congruence subgroups.

For example, in some cases there are isomorphisms between spaces of vector-valued and scalar-valued modular forms. In [2], Bruinier and Bundschuh showed that if $|A| = p$ is an odd prime, then modular forms for ρ^* of weight $k \in \mathbb{Z}$ with $k \equiv \mathrm{sig}(A, Q)/2 \pmod 2$ can be identified with certain modular forms of weight k for $\Gamma_0(p)$ and Nebentypus $\chi_p = \left(\frac{\cdot}{p}\right)$. The isomorphism is given by the component sum

$$\varphi\left(\sum_{\gamma\in A} F_\gamma(\tau)\mathfrak{e}_\gamma\right) = \sum_{\gamma\in A} F_\gamma(p\tau)$$

of the vector-valued modular form $F(\tau) = \sum_{\gamma\in A} F_\gamma(\tau)\mathfrak{e}_\gamma$, where $\mathfrak{e}_\gamma$ denotes the standard basis of the group algebra $\mathbb{C}[A]$. Using similar ideas, Y. Zhang constructed isomorphisms between spaces of vector-valued and scalar-valued modular forms for certain classes of finite quadratic modules which do not necessarily have odd prime order (see [10, 11]).

The condition $k \equiv \mathrm{sig}(A, Q)/2 \pmod 2$ turns out to be crucial for the aforementioned result of Bruinier and Bundschuh, since otherwise the components of any modular form for ρ^* satisfy $F_{-\gamma} = -F_\gamma$ and hence cancel out in pairs in the sum. To obtain a non-zero map in any weight, we twist the component sums of vector-valued modular forms by a Dirichlet character χ mod p with $\chi(-1) = (-1)^{k+\mathrm{sig}(A,Q)/2}$. Suppose that $|A| = p$ with an odd prime p. We define the **twisted component sum** of a modular form $F(\tau) = \sum_{\gamma\in A} F_\gamma(\tau)\mathfrak{e}_\gamma$ for ρ^* by

$$\varphi_\chi\left(\sum_{\gamma\in A} F_\gamma(\tau)\mathfrak{e}_\gamma\right) = \sum_{\gamma\in A} \chi(\gamma)F_\gamma(p\tau),$$

where we fix an identification of A with $\mathbb{Z}/p\mathbb{Z}$ to define $\chi(\gamma)$ for $\gamma \in A$. The assumptions on k and χ imply that φ_χ is not trivially zero. Moreover, we have the following result.

Proposition 1. *The map φ_χ defines an injective homomorphism from the space of modular forms of weight k for ρ^* to the space of scalar-valued modular forms of weight k for $\Gamma_0(p^2)$ with Nebentypus $\chi \otimes \chi_p$.*

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For the proof we refer to Proposition 3 below. It is immediate from the construction that the n-th Fourier coefficient of $\varphi_\chi(F)$ vanishes unless $n \in p(\mathbb{Z} - Q(\gamma))$ for some $\gamma \in A \setminus \{0\}$. However, φ_χ is in general not surjective onto the subspace defined by this vanishing condition. We characterize the image of φ_χ in terms of the Atkin-Lehner involution in Proposition 5 below.

We construct an analogous map φ_χ in the case that $|A| = 2p$ is twice an odd prime p, see Proposition 9; in this case, (A, Q) must have odd signature and all modular forms are of half-integral weight.

As an application, we compute restrictions of Doi-Naganuma lifts of odd weight to components of Hirzebruch-Zagier curves T_ℓ of prime index ℓ. Let $K = \mathbb{Q}(\sqrt{p})$ with a prime $p \equiv 1 \pmod 4$ and let $\mathcal{O}_K$ be its ring of integers. Recall that the Doi-Naganuma lift maps a vector-valued cusp form F of weight k for the dual Weil representation associated to the lattice $(\mathcal{O}_K, -N_{K/\mathbb{Q}})$ to a Hilbert cusp form Φ_F of weight k for $\mathrm{SL}_2(\mathcal{O}_K)$. The restriction of Φ_F to a component of the Hirzebruch-Zagier curve T_ℓ of prime index ℓ is given by the Shimura lift of the vector-valued cusp form of weight $k + 1/2$ for the dual Weil representation of the lattice $(\mathbb{Z}, -\ell x^2)$ obtained by the so-called theta contraction of F as defined in [6], i.e. we have the commutative diagram

$$\begin{tikzcd}
S_k(-N_{K/\mathbb{Q}}) \arrow[r, "\mathrm{Doi-Naganuma}"] \arrow[d, "\text{theta contr.}"] & S_k(\mathrm{SL}_2(\mathcal{O}_K)) \arrow[d, "\text{restr. to } T_\ell"] \\
S_{k+1/2}((-2\ell)) \arrow[r, "\mathrm{Shimura}"] & S_{2k}(\Gamma_0(\ell))
\end{tikzcd}$$

For a proof, see Lemma 10. We show that, on the level of the corresponding scalar-valued modular forms, the theta contraction basically becomes a multiplication by the Jacobi theta function. In this way, passing to scalar-valued modular forms makes it easier to compute the restriction of Φ_F to a component of T_ℓ. To illustrate the result, we consider the case $\ell = p$ in the introduction.

Proposition 2. *Let χ be a Dirichlet character mod p with $\chi(-1) = (-1)^k$. We have the following commutative diagram:*

$$\begin{tikzcd}
S_k(-N_{K/\mathbb{Q}}) \arrow[r, "\varphi_\chi"] \arrow[d, "\text{theta contr.}"] & S_k(\Gamma_0(p^2), \chi \otimes \chi_p) \arrow[d, "G \mapsto \big(G(4p\tau) \cdot \vartheta(\tau)\big)|U_p"] \\
S_{k+1/2}((-2\ell)) \arrow[r, "\varphi_\chi"] & S_{k+1/2}(\Gamma_0(p^2), \chi)
\end{tikzcd}$$

where $\vartheta(\tau) = \sum_{n\in\mathbb{Z}} q^{n^2}$ is the Jacobi theta function and U_p is the usual Hecke operator acting on Fourier expansions by $\big(\sum_n c(n)q^n\big)|U_p = \sum_n c(pn)q^n$.

We refer to Proposition 11 for the general statement and its proof. We also give two numerical examples illustrating the use of the above proposition in Section 4.

The work is organized as follows. We start with preliminaries about modular forms for the Weil representation associated to a finite quadratic module. In Section 3, we investigate twisted component sums of vector-valued modular forms and obtain isomorphisms between spaces of vector-valued and scalar-valued modular forms in the case that the underlying finite quadratic module has order p or $2p$, with an odd prime p. Finally, in Section 4, we explain how these isomorphisms can be used to compute restrictions of Doi-Naganuma lifts of odd weight to components of Hirzebruch-Zagier curves.

2. Modular forms for the Weil representation

A **finite quadratic module** (A, Q) consists of a finite abelian group A and a nondegenerate $\mathbb{Q}/\mathbb{Z}$-valued quadratic form Q on it. The **signature** of (A, Q) is the number $\mathrm{sig}(A, Q) \in \mathbb{Z}/8\mathbb{Z}$ defined through the Gauss sum of A by

$$\mathbf{e}(\mathrm{sig}(A,Q)/8) = \frac{1}{\sqrt{|A|}} \sum_{\gamma \in A} \mathbf{e}(Q(\gamma)), \tag{1}$$

where $\mathbf{e}(x) = e^{2\pi i x}$. By Milgram's formula ([7], appendix 4), this is also the signature mod 8 of any even lattice which induces (A,Q) as its discriminant form.

Let $\mathbb{C}[A]$ be the group algebra of A with basis $\mathfrak{e}_\gamma$, $\gamma \in A$, and let $\mathrm{Mp}_2(\mathbb{Z})$ be the integral metaplectic group, consisting of pairs $\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \pm\sqrt{c\tau + d}\right)$ with $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. The **dual Weil representation** ρ^* is a unitary representation of $\mathrm{Mp}_2(\mathbb{Z})$ on $\mathbb{C}[A]$ which is defined on the generators $S = \left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), \sqrt{\tau}\right)$ and $T = \left(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), 1\right)$ by

$$\rho^*(T)\mathfrak{e}_\gamma = \mathbf{e}(-Q(\gamma))\mathfrak{e}_\gamma, \qquad \rho^*(S)\mathfrak{e}_\gamma = \frac{\mathbf{e}(\mathrm{sig}(A,Q)/8)}{\sqrt{|A|}} \sum_{\beta \in A} \mathbf{e}(\langle \beta, \gamma \rangle)\mathfrak{e}_\beta,$$

where $\langle \beta, \gamma \rangle = Q(\beta + \gamma) - Q(\beta) - Q(\gamma)$ is the bilinear form associated to Q. We also write ρ_A^* if we want to emphasize the dependence on A, or ρ_Λ^* if A is the discriminant form of an even lattice Λ.

A function $F : \mathbb{H} \to \mathbb{C}[A]$ is called a **weakly holomorphic modular form** of weight $k \in \frac{1}{2}\mathbb{Z}$ for ρ^* if it is holomorphic on $\mathbb{H}$, if it satisfies

$$F\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k \rho^* \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \sqrt{c\tau + d}\right) F(\tau)$$

for all $\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \sqrt{c\tau + d}\right) \in \mathrm{Mp}_2(\mathbb{Z})$, and if it is meromorphic at ∞, which means that it has a Fourier expansion of the form

$$F(\tau) = \sum_{\gamma \in A} \sum_{\substack{n \in \mathbb{Z} - Q(\gamma) \\ n \gg -\infty}} c(n,\gamma) q^n \mathfrak{e}_\gamma,$$

with coefficients $c(n,\gamma) \in \mathbb{C}$ and $q = e^{2\pi i \tau}$. Following [2], we will denote the space of all these functions by $A_k(\rho^*)$ (instead of the more common $M_k^!(\rho^*)$). We let $M_k(\rho^*)$ and $S_k(\rho^*)$ be the subspaces of holomorphic modular forms and cusp forms, respectively. If A is the discriminant form of an even lattice Λ, then we also write $A_k(Q)$ or $A_k(\mathbf{S})$ for $A_k(\rho^*)$, where Q is the quadratic form on Λ and $\mathbf{S}$ is the Gram matrix of Q with respect to some basis of Λ.

The element $Z = (-I, i) = S^2$ acts by $\rho^*(Z)\mathfrak{e}_\gamma = (-1)^{\mathrm{sig}(A,Q)/2}\mathfrak{e}_{-\gamma}$ which implies that $A_k(\rho^*) = 0$ if $k + \mathrm{sig}(A,Q)/2$ is not integral, and that the components of any weakly holomorphic modular form $F = \sum_{\gamma \in A} F_\gamma \mathfrak{e}_\gamma \in A_k(\rho^*)$ satisfy

$$F_\gamma = (-1)^{k + \mathrm{sig}(A,Q)/2} F_{-\gamma}$$

for all $\gamma \in A$. Therefore we refer to k as a **symmetric** or **antisymmetric weight** if $k + \mathrm{sig}(A,Q)/2$ is respectively even or odd.

3. Vector-valued and scalar-valued modular forms

In this section, we give isomorphisms between spaces $A_k(\rho^*)$ of vector-valued modular forms for ρ^* and scalar-valued modular forms for $\Gamma_0(p^2)$ and $\Gamma_0(4p^2)$ in the cases $|A| = p$ and $|A| = 2p$ with an odd prime p, for both symmetric and antisymmetric weights $k \in \frac{1}{2}\mathbb{Z}$.

3.1. **Finite quadratic modules of order p.** Suppose that $|A| = p$ is an odd prime. Then $A \cong \mathbb{Z}/p\mathbb{Z}$ with $Q(\gamma) = \alpha\gamma^2/p$ for some $\alpha \in \mathbb{Z}$ with $p \nmid \alpha$. We put $\epsilon = \chi_p(\alpha) = \left(\frac{\alpha}{p}\right)$, and for odd $d \in \mathbb{Z}$ we let

$$\varepsilon_d = \begin{cases} 1, & p \equiv 1 \pmod 4, \\ i, & p \equiv 3 \pmod 4. \end{cases}$$

The evaluation of the quadratic Gauss sum $\sum_{n(p)} \mathbf{e}(\alpha n^2/p) = \chi_p(\alpha)\varepsilon_p\sqrt{p}$ and Milgram's formula (1) show that $\epsilon\varepsilon_p = \mathbf{e}(\mathrm{sig}(A,Q)/8)$. Thus the signature $\mathrm{sig}(A,Q) \in \mathbb{Z}/8\mathbb{Z}$ depends on p and ϵ as shown in the following table:

$p \pmod 4$	1	3
$\epsilon = +1$	0	2
$\epsilon = -1$	4	6

In particular, $\mathrm{sig}(A,Q)$ is even. Hence we can assume that k is an integer since otherwise $A_k(\rho^*) = 0$.

Let χ be a Dirichlet character mod p and let $A_k(p^2, \chi \otimes \chi_p)$ be the space of scalar-valued weakly holomorphic modular forms of weight k for $\Gamma_0(p^2)$ with character $\chi \otimes \chi_p$. We assume that

$$\chi(-1) = (-1)^{k+\mathrm{sig}(A,Q)/2} \tag{2}$$

since otherwise $A_k(p^2, \chi \otimes \chi_p) = 0$. We define the subspace

$$A_k^\epsilon(p^2, \chi \otimes \chi_p) = \left\{ \sum_{\substack{n \in \mathbb{Z} \\ n \gg -\infty}} c(n) q^n \in A_k(p^2, \chi \otimes \chi_p) \ : \ \begin{array}{l} c(n) = 0 \text{ unless } n \in p(\mathbb{Z} - Q(\gamma)) \\ \text{for some } \gamma \in (\mathbb{Z}/p\mathbb{Z})^* \end{array} \right\}.$$

The condition in the brackets can be restated by saying that $c(n) = 0$ unless $\chi_p(-n) = \epsilon$. Hence we also call it the **ϵ-condition**. Note that, in contrast to the definition of the ϵ-condition in [2], we also require that $c(n) = 0$ if $p \mid n$.

We define the **twisted component sum** of a vector-valued modular form $F(\tau) = \sum_{\gamma(p)} F_\gamma(\tau)\mathfrak{e}_\gamma \in A_k(\rho^*)$ by

$$\varphi_\chi\left(\sum_{\gamma(p)} F_\gamma(\tau)\mathfrak{e}_\gamma\right) = \sum_{\gamma(p)^*} \chi(\gamma) F_\gamma(p\tau).$$

The parity condition (2) ensures that $\varphi_\chi(F)$ is not trivially identically zero. Again, also for symmetric weight k and with χ the trivial character mod p, our twisted component sum differs from the component sum $\sum_{\gamma(p)} F_\gamma(p\tau)$ considered in [2] since we omit the zero component $F_0(p\tau)$ in the sum. For antisymmetric weight k we have $F_0 = 0$.

Proposition 3. *If $F \in A_k(\rho^*)$, then $\varphi_\chi(F) \in A_k^\epsilon(p^2, \chi \otimes \chi_p)$. Furthermore, φ_χ is injective.*

Proof. Let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(p^2)$ and $N = \left(\begin{smallmatrix} a & bp \\ c/p & d \end{smallmatrix}\right)$. We write

$$\varphi_\chi(F)(M\tau) = \sum_{\gamma(p)^*} \chi(\gamma) F_\gamma(p \cdot M\tau) = \sum_{\gamma(p)^*} \chi(\gamma) F_\gamma(N(p\tau)).$$

By [1], Theorem 5.2, we have $\rho^*(N)\mathfrak{e}_\gamma = \chi_p(d)\mathfrak{e}_{d\gamma}$, hence

$$F_\gamma(N(p\tau)) = \chi_p(d)(c\tau + d)^k F_{d^{-1}\gamma}(p\tau),$$

where d^{-1} denotes an inverse of d mod p. Thus we find

$$\varphi_\chi(F)(M\tau) = \chi_p(d)(c\tau+d)^k \sum_{\gamma(p)^*} \chi(\gamma) F_{d^{-1}\gamma}(p\tau) = \chi\chi_p(d)(c\tau+d)^k \varphi_\chi(F)(\tau).$$

It is clear that $\varphi_\chi(F)$ is holomorphic on $\mathbb{H}$ and meromorphic at the cusps (since $F_\gamma|_k M$ is a linear combination of components of F), and that it satisfies the ϵ-condition, so $\varphi_\chi(F) \in A_k^\epsilon(p^2, \chi \otimes \chi_p)$.

Now suppose that $\varphi_\chi(F) = 0$ for some $F \in A_k(\rho^*)$. Since the components F_γ, F_β for $\beta \neq \pm\gamma$ are supported on disjoint index sets, $\varphi_\chi(F) = 0$ implies that $F_\gamma = 0$ for all $\gamma \neq 0$. But then the zero component of F satisfies

$$F_0|_k S = \frac{\mathbf{e}(\mathrm{sig}(A,Q)/8)}{\sqrt{p}} \sum_{\gamma(p)} F_\gamma = \frac{\mathbf{e}(\mathrm{sig}(A,Q)/8)}{\sqrt{p}} F_0,$$

and applying $|_k S$ a second time, we find $(-1)^k F_0 = \frac{\mathbf{e}(\mathrm{sig}(A,Q)/4)}{p} F_0$, hence $F_0 = 0$. We have shown that $F = 0$, so φ_χ is injective. $\square$

The map φ_χ is in general not surjective. Its image can be described in terms of the behaviour of certain twists of G under the Atkin-Lehner involution, which we explain now.

We can split $G(\tau) = \sum_{n \gg -\infty} c(n)q^n \in A_k^\epsilon(p^2, \chi \otimes \chi_p)$ into components

$$G(\tau) = \sum_{\gamma(p)^*} G_\gamma(\tau), \qquad G_\gamma(\tau) = \frac{1}{2} \sum_{\substack{n \in p(\mathbb{Z} - Q(\gamma)) \\ n \gg -\infty}} c(n) q^n.$$

We define the **component-wise twist** of G by a Dirichlet character ψ mod p by

$$G_\psi(\tau) = \sum_{\gamma(p)^*} \psi(\gamma) G_\gamma(\tau).$$

Note that the component-wise twist differs from the usual twist $\sum_{n \gg -\infty} \psi(n)c(n)q^n$ of a modular form.

Lemma 4. *Let $G \in A_k^\epsilon(p^2, \chi \otimes \chi_p)$ and let ψ be a Dirichlet character mod p. Then $G_\psi \in A_k^\epsilon(p^2, \psi \otimes \chi \otimes \chi_p)$.*

Proof. We can write

$$G_\psi = \frac{1}{2p} \sum_{\gamma(p)^*} \psi(\gamma) \sum_{j(p)} G\Big|_k \begin{pmatrix} 1 & j/p \\ 0 & 1 \end{pmatrix} \mathbf{e}(jQ(\gamma)).$$

Let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(p^2)$. We compute

$$\begin{pmatrix} 1 & j/p \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & -d^2 j/p \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a + cj/p & b + (1-ad)dj/p - d^2cj^2/p^2 \\ c & d - d^2cj/p \end{pmatrix} \in \Gamma_0(p^2).$$

Note that the d-entry of this matrix equals d mod p. Hence we obtain

$$G_\psi|_k M = \chi\chi_p(d) \frac{1}{2p} \sum_{\gamma \in (p)^*} \psi(\gamma) \sum_{j(p)} G\Big|_k \begin{pmatrix} 1 & d^2j/p \\ 0 & 1 \end{pmatrix} \mathbf{e}(jQ(\gamma)).$$

Replacing d^2j by j and then γ by $d\gamma$ gives a factor $\psi(d)$ and completes the proof. $\square$

We let $W_{p^2} = \left(\begin{smallmatrix} 0 & -1 \\ p^2 & 0 \end{smallmatrix}\right)$ be the Atkin-Lehner (or Fricke) involution. It maps $A_k(p^2, \chi \otimes \chi_p)$ to $A_k(p^2, \overline{\chi} \otimes \chi_p)$, but it does in general not respect the ϵ-condition. We say that $G \in A_k^\epsilon(p^2, \chi \otimes \chi_p)$ satisfies the **Atkin-Lehner condition** if the twists of G by all Dirichlet characters ψ mod p with $\psi \neq \overline{\chi}$ satisfy

$$G_\psi|_k W_{p^2} = \overline{\psi\chi(2\alpha)} \frac{g(\psi\chi)}{\sqrt{p}} \mathbf{e}(\mathrm{sig}(A,Q)/8) G_{\overline{\psi}\overline{\chi}^2}, \tag{3}$$

where $g(\psi\chi) = \sum_{n(p)^*} \psi\chi(n)\mathbf{e}(n/p)$ is a Gauss sum. We let $A_k^{\epsilon,\mathrm{AL}}(p^2, \chi \otimes \chi_p)$ be the subspace of $A_k^\epsilon(p^2, \chi \otimes \chi_p)$ consisting of all forms satisfying the Atkin-Lehner condition.

Proposition 5. *The linear map*

$$\varphi_\chi : A_k(\rho^*) \to A_k^{\epsilon,\mathrm{AL}}(p^2, \chi \otimes \chi_p), \qquad F(\tau) = \sum_{\gamma(p)} F_\gamma(\tau)\mathfrak{e}_\gamma \mapsto \sum_{\gamma(p)^*} \chi(\gamma) F_\gamma(p\tau),$$

is an isomorphism. The inverse map is given by

$$\varphi_\chi^{-1} : G(\tau) = \sum_{\gamma(p)^*} G_\gamma(\tau) \mapsto \sum_{\gamma(p)^*} \overline{\chi(\gamma)} G_\gamma(\tau/p)\mathfrak{e}_\gamma + G_0(\tau/p)\mathfrak{e}_0,$$

where G_0 is defined by

$$G_0(\tau) = \frac{\sqrt{p}}{p-1} \mathbf{e}(-\mathrm{sig}(A,Q)/8) \left(G_{\overline{\chi}}|_k W_{p^2}\right)(\tau) + \frac{1}{p-1} G_{\overline{\chi}}(\tau).$$

Proof. We first show that $G = \varphi_\chi(F)$ for $F = \sum_{\gamma(p)} F_\gamma \mathfrak{e}_\gamma \in A_k(\rho^*)$ satisfies the Atkin-Lehner condition. Let ψ be a Dirichlet character mod p and let $\delta_{\psi,\overline{\chi}} = 1$ if $\psi = \overline{\chi}$ and $\delta_{\psi,\overline{\chi}} = 0$ otherwise. We compute

$$\begin{aligned}
\left(G_\psi|_k W_{p^2}\right)(\tau) &= p^k (p^2\tau)^{-k} G_\psi\left(-\frac{1}{p^2\tau}\right) \\
&= \sum_{\gamma(p)^*} \psi(\gamma)\chi(\gamma)(p\tau)^{-k} F_\gamma\left(-\frac{1}{p\tau}\right) \\
&= \sum_{\gamma(p)^*} \psi(\gamma)\chi(\gamma) \frac{\mathbf{e}(\mathrm{sig}(A,Q)/8)}{\sqrt{p}} \sum_{\beta(p)} \mathbf{e}((\beta,\gamma)) F_\beta(p\tau) \\
&= \frac{\mathbf{e}(\mathrm{sig}(A,Q)/8)}{\sqrt{p}} \left(\sum_{\beta(p)^*} \left(\sum_{\gamma(p)^*} \psi(\gamma)\chi(\gamma)\mathbf{e}(2\alpha\beta\gamma/p) \right) F_\beta(p\tau) + \delta_{\psi,\overline{\chi}}(p-1)F_0(p\tau) \right) \\
&= \frac{\mathbf{e}(\mathrm{sig}(A,Q)/8)}{\sqrt{p}} \left(\sum_{\gamma(p)^*} \psi(\gamma)\chi(\gamma)\mathbf{e}(2\alpha\gamma/p) \sum_{\beta(p)^*} \overline{\psi(\beta)\chi(\beta)} F_\beta(p\tau) + \delta_{\psi,\overline{\chi}}(p-1)F_0(p\tau) \right) \\
&= \frac{\mathbf{e}(\mathrm{sig}(A,Q)/8)}{\sqrt{p}} \left(\overline{\psi\chi(2\alpha)} g(\psi\chi) G_{\overline{\psi}\overline{\chi}^2}(\tau) + \delta_{\psi,\overline{\chi}}(p-1)F_0(p\tau) \right).
\end{aligned}$$

This shows that G satisfies the Atkin-Lehner condition, i.e., $G \in A_k^{\epsilon,\mathrm{AL}}(p^2, \chi \otimes \chi_p)$.

Conversely, if $G \in A_k^{\epsilon,\mathrm{AL}}(p^2, \chi \otimes \chi_p)$ and if G_0 is defined as in the proposition, then we can reverse the above computation (with $F_0(p\tau) = G_0(\tau)$ and $F_\gamma(p\tau) = \overline{\chi(\gamma)} G_\gamma(\tau)$ for $\gamma \neq 0$) to see that the second and third line agree for all Dirichlet characters ψ mod p. By character orthogonality, we obtain that

$$F_\gamma|_k S = \frac{\mathbf{e}(\mathrm{sig}(A,Q)/8)}{\sqrt{p}} \sum_{\beta(p)} \mathbf{e}((\beta,\gamma)) F_\beta$$

for all $\gamma \neq 0$. A short computation shows that this equation also implies

$$F_0|_k S = \frac{\mathbf{e}(\mathrm{sig}(A,Q)/8)}{\sqrt{p}} \sum_{\beta(p)} F_\beta.$$

Furthermore, it is easy to check that F_γ transforms correctly under T. We find that $\varphi_\chi^{-1}(G) \in A_k(\rho^*)$. Since $\varphi_\chi \circ \varphi_\chi^{-1} = \mathrm{id}$ and φ_χ is injective, φ_χ is an isomorphism. $\square$

3.2. **Finite quadratic modules of order** $2p$. Suppose that $|A| = 2p$ with an odd prime p. Then $A \cong \mathbb{Z}/2p\mathbb{Z}$ with the quadratic form

$$Q(p\gamma_1 + \gamma_2) = \delta\gamma_1^2/4 + \alpha\gamma_2^2/p$$

for $\gamma_1 \in \mathbb{Z}/2\mathbb{Z}$ and $\gamma_2 \in \mathbb{Z}/p\mathbb{Z}$, where $\delta \in \{\pm 1\}$ and $\alpha \in \mathbb{Z}$ with $p \nmid \alpha$. Set $\epsilon = \chi_p(\alpha)$. Using the quadratic Gauss sum and Milgram's formula we obtain $(1+\delta i)\epsilon\varepsilon_p = \sqrt{2}\mathbf{e}(\mathrm{sig}(A,Q)/8)$, so the signature $\mathrm{sig}(A,Q) \in \mathbb{Z}/8\mathbb{Z}$ is given in terms of p, ϵ and δ as follows:

$p \pmod 4$	1	3
$\epsilon = +1$	δ	$\delta + 2$
$\epsilon = -1$	$\delta + 4$	$\delta + 6$

Now $\mathrm{sig}(A,Q)$ is odd. Hence we can assume that k is half-integral since otherwise $A_k(\rho^*) = 0$.

Let us briefly recall the definition of modular forms of half-integral weight. The theta multiplier is given by

$$\nu_\vartheta(M) = \left(\frac{c}{d}\right)\varepsilon_d^{-1}, \qquad \varepsilon_d = \left(\frac{2}{d}\right)\mathbf{e}((1-d)/8) = \begin{cases} 1, & d \equiv 1(4), \\ i, & d \equiv 3(4), \end{cases}$$

for $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4)$. A function $G : \mathbb{H} \to \mathbb{C}$ is called a weakly holomorphic modular form of weight $k \in \frac{1}{2} + \mathbb{Z}$ for $\Gamma_0(N)$ with $4 \mid N$ and character χ mod N if it is holomorphic on $\mathbb{H}$ and meromorphic at the cusps, and if it transforms as

$$G(M\tau) = \chi(M)\nu_\vartheta(M)^{\pm 1}(c\tau+d)^k G(\tau)$$

for $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$, where the sign in $\nu_\vartheta(M)^{\pm 1}$ is chosen such that $\nu_\vartheta(-1)^{\pm 1}\chi(-1)i^{2k} = 1$. We denote the space of all these functions by $A_k(N,\chi)$.

We let χ be a character mod p and we again assume that

$$\chi(-1) = (-1)^{k+\mathrm{sig}(A,Q)/2}.$$

We consider the space

$$A_k^{\epsilon}(16p^2,\chi) = \left\{ \sum_{n \gg -\infty} c(n)q^n \in A_k(16p^2,\chi) \ : \ \begin{array}{l} c(n) = 0 \text{ unless } n \in 4p(\mathbb{Z} - Q(\gamma)) \\ \text{for some } \gamma \in (\mathbb{Z}/2p\mathbb{Z}) \setminus \{0,p\} \end{array} \right\}$$

and we let $A_k^{\epsilon}(4p^2,\chi) = A_k^{\epsilon}(16p^2,\chi) \cap A_k(4p^2,\chi)$.

We define the **twisted component sum** of $F(\tau) = \sum_{\gamma(2p)} F_\gamma(\tau)\mathfrak{e}_\gamma \in A_k(\rho^*)$ by

$$\varphi_\chi\left(\sum_{\gamma(2p)} F_\gamma(\tau)\mathfrak{e}_\gamma\right) = \sum_{\gamma(2p)} \chi(\gamma)F_\gamma(4p\tau).$$

Note that, since χ is a character mod p, we discard the components F_0 and F_p in the twisted component sum. If k is an antisymmetric weight, then $F_0 = F_p = 0$ is automatic. This map was already suggested in [3], p. 70, in the context of Jacobi forms.

Proposition 6. *If $F \in A_k(\rho^*)$, then $\varphi_\chi(F) \in A_k^{\epsilon}(4p^2,\chi)$. Furthermore, φ_χ is injective.*

Proof. We first show that that $\varphi_\chi(F)$ transforms correctly under $\Gamma_0(16p^2)$. For $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(16p^2)$ we let $N = \left(\begin{smallmatrix} a & 4pb \\ c/4p & d \end{smallmatrix}\right)$. Then we compute

$$\varphi_\chi(F)(M\tau) = \sum_{\gamma(2p)} \chi(\gamma)F_\gamma(4p\cdot M\tau) = \sum_{\gamma(2p)} \chi(\gamma)F_\gamma(N(4p\tau)).$$

Using [1], Theorem 5.2, we obtain

$$F_\gamma(N(4p\tau)) = \left(\frac{c/4p}{d}\right)\left(\frac{d}{2p}\right)\mathbf{e}((1-d)\delta/8)(c\tau+d)^k F_{d^{-1}\gamma}(4p\tau).$$

We compute

$$\left(\frac{c/4p}{d}\right)\left(\frac{d}{2p}\right)\mathbf{e}((1-d)\delta/8) = \left(\frac{p}{d}\right)\left(\frac{d}{p}\right)\left(\frac{c}{d}\right)\left(\frac{2}{d}\right)\mathbf{e}((1-d)\delta/8) = \left(\frac{p}{d}\right)\left(\frac{d}{p}\right)\nu_\vartheta(M)^{-\delta}.$$

By quadratic reciprocity we have $\left(\frac{p}{d}\right)\left(\frac{d}{p}\right) = 1$ if $p \equiv 1 \pmod 4$ and

$$\left(\frac{p}{d}\right)\left(\frac{d}{p}\right) = \begin{cases} 1, & \text{if } d \equiv 1 \pmod 4, \\ -1, & \text{if } d \equiv 3 \pmod 4, \end{cases}$$

if $p \equiv 3 \pmod 4$. This gives the stated transformation behaviour under $\Gamma_0(16p^2)$.

In order to show the transformation behaviour under $\Gamma_0(4p^2)$, it suffices to check the transformation under the matrices $U_{4p^2j} = \left(\begin{smallmatrix} 1 & 0 \\ 4p^2j & 1 \end{smallmatrix}\right)$ for $j = 0,1,2,3$ since they represent $\Gamma_0(16p^2)\backslash\Gamma_0(4p^2)$. We compute

$$\begin{aligned}
\varphi_\chi(F)(U_{4p^2j}\tau) &= \sum_{\gamma(2p)} \chi(\gamma)F_\gamma(4p\cdot U_{4p^2j}\tau) \\
&= \sum_{\gamma(2p)} \chi(\gamma)F_\gamma(U_{pj}(4p\tau)) \\
&= \sum_{\gamma(2p)} \chi(\gamma)F_\gamma(S^{-1}T^{-pj}S(4p\tau)) \\
&= (4p^2j\tau+1)^k\frac{1}{2p}\sum_{\gamma(2p)}\chi(\gamma)\sum_{\beta(2p)}\mathbf{e}(-(\beta,\gamma))\mathbf{e}(jpQ(\beta))\sum_{\mu(2p)}\mathbf{e}((\mu,\beta))F_\mu(4p\tau).
\end{aligned}$$

If we write $\gamma = p\gamma_1 + \gamma_2$ with $\gamma_1 \in \mathbb{Z}/2\mathbb{Z}$ and $\gamma_2 \in \mathbb{Z}/p\mathbb{Z}$, and similarly for β, and use that χ only depends on γ_2, we see that the sum over γ_1 vanishes unless $\beta_2 = 0$. This means that we can replace $\mathbf{e}(jpQ(\beta))$ by 1 in the above sum. But then the sum over β equals $2p$ if $\mu = \gamma$, and vanishes otherwise. Hence we get

$$\varphi_\chi(F)(U_{4p^2j}\tau) = (4p^2j\tau + 1)^k \sum_{\gamma(2p)} \chi(\gamma)F_\gamma(4p\tau) = (4p^2j\tau + 1)^k \varphi_\chi(F)(\tau).$$

This shows that $\varphi_\chi(F)$ transforms correctly under $\Gamma_0(4p^2)$. It is easy to see that φ_χ is holomorphic on $\mathbb{H}$ and meromorphic at the cusps, and that it satisfies the ϵ-condition.

Now suppose that $\varphi_\chi(F) = 0$ for some $F \in A_k(\rho^*)$. By comparing the index sets on which the components F_γ are supported, we obtain that $F_\gamma = 0$ for $\gamma \notin \{0, p\}$. Then the transformation behaviour of F under S implies

$$F_0(-1/\tau) = \tau^k \frac{\mathbf{e}(\mathrm{sig}(A,Q)/8)}{\sqrt{2p}}(F_0(\tau) + F_p(\tau)), \qquad F_p(-1/\tau) = \tau^k \frac{\mathbf{e}(\mathrm{sig}(A,Q)/8)}{\sqrt{2p}}(F_0(\tau) - F_p(\tau)).$$

Applying $\tau \mapsto -1/\tau$ a second time, we get $i^{2k}F_0 = \frac{\mathbf{e}(\mathrm{sig}(A,Q)/4)}{p}F_0$ and $i^{2k}F_p = \frac{\mathbf{e}(\mathrm{sig}(A,Q)/4)}{p}F_p$, hence $F_0 = F_p = 0$. Thus $F = 0$ and φ_χ is injective. $\square$

We split $G = \sum_{n \gg -\infty} c(n)q^n \in A_k^\epsilon(16p^2, \chi)$ into components

$$G(\tau) = \sum_{\gamma(2p)} G_\gamma(\tau), \qquad G_\gamma(\tau) = \frac{1}{2} \sum_{\substack{n \in 4p(\mathbb{Z} - Q(\gamma)) \\ n \gg -\infty}} c(n)q^n,$$

and define its **component-wise twist** by a Dirichlet character ψ mod p by

$$G_\psi(\tau) = \sum_{\gamma(2p)} \psi(\gamma)G_\gamma(\tau).$$

Lemma 7. *Let $G \in A_k^\epsilon(16p^2, \chi)$ and let ψ be a Dirichlet character mod p. Then $G_\psi \in A_k^\epsilon(16p^2, \psi \otimes \chi)$.*

Proof. The proof is analogous to the proof of Lemma 4, so we leave the details to the reader. $\square$

In contrast to the case $|A| = p$ we need another notion to describe the image of φ_χ. We call $\gamma \in \mathbb{Z}/2p\mathbb{Z}$ even if $4pQ(\gamma)$ is even, and odd if $4pQ(\gamma)$ is odd. The **even** and **odd parts** of $G \in A_k^\epsilon(16p^2, \chi)$ are defined by

$$G^{\mathrm{even}}(\tau) = \sum_{\substack{\gamma(2p) \\ \gamma \text{ even}}} G_\gamma(\tau), \qquad G^{\mathrm{odd}}(\tau) = \sum_{\substack{\gamma(2p) \\ \gamma \text{ odd}}} G_\gamma(\tau).$$

Note that taking the even and odd parts of G commutes with component-wise twisting.

Lemma 8. *If $G \in A_k^\epsilon(16p^2, \chi)$ then $G^{\mathrm{even}}, G^{\mathrm{odd}} \in A_k^\epsilon(16p^2, \chi)$ as well.*

Proof. We can write

$$G^{\mathrm{even}}(\tau) = \frac{1}{2}\left(G(\tau) + G(\tau + 1/2)\right), \qquad G^{\mathrm{odd}}(\tau) = \frac{1}{2}\left(G(\tau) - G(\tau + 1/2)\right).$$

For $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(16p^2)$ we have

$$\begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & -1/2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a + c/2 & b + (d-a)/2 - c/4 \\ c & d - c/2 \end{pmatrix} \in \Gamma_0(16p^2),$$

which easily implies $G(\tau + 1/2) \in A_k^\epsilon(16p^2, \chi)$. This proves the lemma. $\square$

For $G \in A_k(16p^2, \chi)$ we define the Atkin-Lehner involution

$$G|_k W_{16p^2} = (4p)^{-k}(-i\tau)^{-k} G(-1/16p^2\tau).$$

Then $G|_k W_{16p^2} \in A_k(16p^2, \overline{\chi})$ and $G|_k W_{16p^2}|_k W_{16p^2} = G$. In general, the Atkin-Lehner involution does not preserve the ϵ-condition. We say that $G \in A_k^{\epsilon}(16p^2, \chi)$ satisfies the **Atkin-Lehner condition** if its twists by all Dirichlet characters ψ mod p with $\psi \neq \overline{\chi}$ satisfy

$$G_\psi|_k W_{16p^2} = \overline{\psi\chi(2\alpha)} \frac{\sqrt{2}g(\psi\chi)}{\sqrt{p}} \mathbf{e}(\mathrm{sig}(A,Q)/8) i^k G^{\mathrm{even}}_{\overline{\psi}\overline{\chi}^2},$$

where $g(\psi\chi) = \sum_{n(p)^*} \psi\chi(n)\mathbf{e}(n/p)$ is a Gauss sum. Let $A_k^{\epsilon,\mathrm{AL}}(16p^2,\chi)$ be the subspace of $A_k^{\epsilon}(16p^2,\chi)$ satisfying the Atkin-Lehner condition. Note that, after applying W_{16p^2} and a short calculation, the Atkin-Lehner condition also implies that

$$(G^{\mathrm{even}}_\psi - G^{\mathrm{odd}}_\psi)|_k W_{16p^2} = \overline{\psi\chi(2\alpha)} \frac{\sqrt{2}g(\psi\chi)}{\sqrt{p}} \mathbf{e}(\mathrm{sig}(A,Q)/8) i^k G^{\mathrm{odd}}_{\overline{\psi}\overline{\chi}^2}$$

for $\psi \neq \overline{\chi}$.

Proposition 9. *The linear map*

$$\varphi_\chi : A_k(\rho^*) \to A_k^{\epsilon,\mathrm{AL}}(16p^2,\chi), \qquad F(\tau) = \sum_{\gamma(2p)} F_\gamma(\tau)\mathfrak{e}_\gamma \mapsto \sum_{\gamma(2p)} \chi(\gamma) F_\gamma(4p\tau),$$

is an isomorphism. The inverse map is given by

$$\varphi_\chi^{-1} : G(\tau) = \sum_{\gamma(2p)} G_\gamma(\tau) \mapsto \sum_{\substack{\gamma(2p) \\ \gamma\neq 0,p(2p)}} \overline{\chi(\gamma)} G_\gamma(\tau/4p)\mathfrak{e}_\gamma + G_0(\tau/4p)\mathfrak{e}_0 + G_p(\tau/4p)\mathfrak{e}_p,$$

where G_0 and G_p are defined by

$$G_0(\tau) = \frac{\sqrt{p}}{\sqrt{2}(p-1)} \mathbf{e}(-\mathrm{sig}(A,Q)/8) i^{-k} (G^{\mathrm{even}}_{\overline{\chi}}(\tau) + G^{\mathrm{odd}}_{\overline{\chi}}(\tau))|_k W_{16p^2} + \frac{1}{p-1} G^{\mathrm{even}}_{\overline{\chi}}(\tau),$$
$$G_p(\tau) = \frac{\sqrt{p}}{\sqrt{2}(p-1)} \mathbf{e}(-\mathrm{sig}(A,Q)/8) i^{-k} (G^{\mathrm{even}}_{\overline{\chi}}(\tau) - G^{\mathrm{odd}}_{\overline{\chi}}(\tau))|_k W_{16p^2} + \frac{1}{p-1} G^{\mathrm{odd}}_{\overline{\chi}}(\tau).$$

Proof. The proof is very similar to the proof of Proposition 5, so we omit it for brevity. □

4. Application: the Doi-Naganuma lift and theta contraction

Let $K = \mathbb{Q}(\sqrt{p})$ for a prime $p \equiv 1 \pmod 4$ and let $\mathcal{O}_K$ be its ring of integers. Let $\mathcal{O}_K^{\#} = (1/\sqrt{p})\mathcal{O}_K$ be the dual lattice of $\mathcal{O}_K$ with respect to the trace and consider the finite quadratic module $(\mathcal{O}_K^{\#}/\mathcal{O}_K, -N_{K/\mathbb{Q}})$. It has order p and signature 0 mod 8. Let $\lambda \in \mathcal{O}_K$ be any totally positive prime with norm $\ell = N_{K/\mathbb{Q}}(\lambda)$, and let $b \in \mathbb{Z}$ be any integer with $b^2 \equiv p \pmod{4\ell}$ (which exists by quadratic reciprocity). Since λ is prime, one of $(b+\sqrt{p})/2\lambda, (b-\sqrt{p})/2\lambda$ is integral; it is then straightforward to show that $\{\lambda', (b \pm \sqrt{p})/2\lambda\}$ is a $\mathbb{Z}$-basis of $\mathcal{O}_K$ and

$$\mathbf{S} = \begin{pmatrix} -2\ell & -b \\ -b & \frac{p-b^2}{2\ell} \end{pmatrix}$$

is the Gram matrix of $-N_{K/\mathbb{Q}}$ in that basis.

If $\ell \neq p$, then for each $r \in \mathbb{Z}/p\mathbb{Z}$ there exists a unique $a \in \mathbb{Z}/2\ell\mathbb{Z}$ with $r \equiv ab \pmod{2\ell}$. We fix a bijection $\mathcal{O}_K^{\#}/\mathcal{O}_K \cong \mathbb{Z}/p\mathbb{Z}$ by sending $r \in \mathbb{Z}/p\mathbb{Z}$ to the element

$$\gamma_{a,r} + \mathcal{O}_K = \frac{a \pm r/\sqrt{p}}{2\lambda} + \mathcal{O}_K \in \mathcal{O}_K^{\#}/\mathcal{O}_K.$$

If $\ell = p$, then we fix the bijection which identifies $a \in \mathbb{Z}/p\mathbb{Z}$ with $\gamma_a + \mathcal{O}_K = \frac{a(1+\sqrt{p})}{2\lambda} + \mathcal{O}_K$ instead. (In this case, one can always take $b = p$ and $\lambda = \varepsilon\sqrt{p}$ if ε is the fundamental unit of $\mathcal{O}_K$.)

The **theta decomposition** identifies vector-valued modular forms for the dual Weil representation attached to $(\mathcal{O}_K^{\#}/\mathcal{O}_K, -N_{K/\mathbb{Q}})$ with vector-valued Jacobi forms of fractional index $p/4\ell$ for the dual Weil representation attached to the discriminant form with Gram matrix (-2ℓ) and a particular representation of the Heisenberg group, see e.g. [9]. By setting the Heisenberg variable of those Jacobi forms equal to zero one obtains the **theta contraction**, a graded homomorphism between the modular forms $M_*(-N_{K/\mathbb{Q}})$ and $M_{*+1/2}((-2\ell))$ as graded modules over the ring $M_*(\mathrm{SL}_2(\mathbb{Z}))$ of scalar-valued modular forms. This was introduced by Ma [6] in order to study the quasi-pullback of Borcherds products. Explicitly in terms of Fourier coefficients, it is the map

$$\Theta : M_k(-N_{K/\mathbb{Q}}) \to M_{k+1/2}((-2\ell)), \quad \sum_{\gamma \in \mathcal{O}_K^\#/\mathcal{O}_K} \sum_{n \in \mathbb{Z}+N_{K/\mathbb{Q}}(\gamma)} c(n,\gamma) q^n \mathfrak{e}_\gamma \mapsto \sum_{a \in \mathbb{Z}/2\ell\mathbb{Z}} \sum_{n \in \mathbb{Z}+a^2/4\ell} \tilde{c}(n,a) q^n \mathfrak{e}_a,$$

where

$$\tilde{c}(n,a) = \sum_{r \equiv ab\,(2\ell)} c\left(n - \frac{r^2}{4\ell p}, \gamma_{a,r}\right).$$

Possibly the most important aspect of the theta contraction (which can be defined more generally) is that it fits into a commutative diagram involving the additive theta lift (of Oda and Rallis-Schiffmann) and restriction to Heegner divisors: letting Λ be an even lattice of type $(2,b^-)$ such that $b^- \geq 2$ is greater than the Witt rank of Λ, and $\lambda^\perp$ the orthogonal complement of a primitive, negative-norm vector $\lambda \in \Lambda$, the natural pullback map Res for orthogonal modular forms satisfies

$$\begin{array}{ccc} S_{k+1-b^-/2}(\rho_\Lambda^*) & \xrightarrow{\text{Theta lift}} & S_k(O(\Lambda)) \\ \downarrow \Theta & & \downarrow \text{Res} \\ S_{k+1-(b^--1)/2}(\rho_{\lambda^\perp}^*) & \xrightarrow{\text{Theta lift}} & S_k(O(\lambda^\perp)) \end{array}$$

for $k \geq 2$.

For the Doi-Naganuma lift (i.e. $b^- = 2$) this can be made very explicit. Recall that for a cusp form

$$F(\tau) = \sum_{\gamma \in \mathcal{O}_K^\#/\mathcal{O}_K} \sum_{n \in \mathbb{Z}+N_{K/\mathbb{Q}}(\gamma)} c(n,\gamma) q^n \mathfrak{e}_\gamma \in S_k(-N_{K/\mathbb{Q}})$$

of weight $k \geq 2$, the Doi-Naganuma lift is a Hilbert cusp form Φ_F of weight k for $\mathrm{SL}_2(\mathcal{O}_K)$ with Fourier expansion

$$\Phi_F(\tau_1,\tau_2) = \sum_{\substack{\nu \in \mathcal{O}_K^\# \\ \nu,\nu'>0}} \sum_{n=1}^\infty c(\nu\nu',\nu) n^{k-1} \mathbf{q}^{n\nu}, \quad \text{where } \mathbf{q}^\nu = e^{2\pi i(\nu\tau_1+\nu'\tau_2)}.$$

Moreover Φ_F satisfies the graded symmetry $\Phi_F(\tau_1,\tau_2) = (-1)^k \Phi_F(\tau_2,\tau_1)$.

With λ as above, there is a natural restriction map onto a component of the Hirzebruch-Zagier curve T_ℓ:

$$\text{Res} : S_k(\mathrm{SL}_2(\mathcal{O}_K)) \to S_{2k}(\Gamma_0(\ell)), \quad f(\tau_1,\tau_2) \mapsto f(\lambda\tau, \lambda'\tau).$$

It turns out that $\text{Res}(\Phi_F)$ equals the Shimura lift

$$\sum_{a=1}^\infty \sum_{n=1}^\infty \tilde{c}(a^2/4\ell, a) n^{k-1} q^{na}$$

of the contracted form $\Theta F = \sum_{a,n} \tilde{c}(n,a) q^n \mathfrak{e}_a \in S_{k+1/2}((-2\ell))$.

Lemma 10. *We have the following commutative diagram:*

$$\begin{array}{ccc} S_k(-N_{K/\mathbb{Q}}) & \xrightarrow{\text{Doi}-\text{Naganuma}} & S_k(\mathrm{SL}_2(\mathcal{O}_K)) \\ \downarrow \Theta & & \downarrow \text{Res} \\ S_{k+1/2}((-2\ell)) & \xrightarrow{\text{Shimura}} & S_{2k}(\Gamma_0(\ell)) \end{array}$$

Proof. Since the elements $\nu \in \mathcal{O}_K^\#$ with $\mathrm{Tr}(\nu\lambda) = a \in \mathbb{N}$ are exactly those of the form $\gamma_{a,r} = \frac{a \pm r/\sqrt{p}}{2\lambda}$ with $r \equiv ab \bmod 2\ell$, we find

$$\begin{aligned}\Phi_F(\lambda\tau, \lambda'\tau) &= \sum_{n=1}^\infty \sum_{\mathrm{Tr}(\nu\lambda)=a} c(\nu\nu', \nu) n^{k-1} q^{na} \\ &= \sum_{n=1}^\infty \sum_{a=1}^\infty \sum_{r \equiv ab\,(2\ell)} c\left(\frac{a^2}{4\ell} - \frac{r^2}{4\ell p}, \frac{a \pm r/\sqrt{p}}{2\lambda}\right) n^{k-1} q^{na} \\ &= \sum_{n,a=1}^\infty \tilde{c}(a^2/4\ell, a) n^{k-1} q^{na},\end{aligned}$$

i.e. $\mathrm{Res}(\Phi_F)(\tau) = \Phi_F(\lambda\tau, \lambda'\tau)$ is the Shimura lift of the contracted form $\Theta F \in S_{k+1/2}((-2\ell))$. $\square$

In this section we observe that this relationship takes a simple form in terms of twisted component sums of F and ΘF. Recall that for a q-series $f(\tau) = \sum_n c(n) q^n$ the Hecke operator U_p is defined by

$$f|U_p(\tau) = \sum_n c(pn) q^n = \sum_{n \equiv 0\,(p)} c(n) q^{n/p}.$$

Proposition 11.

(i) *Suppose $\ell \neq p$. Let ψ_ℓ and ψ_p be Dirichlet characters modulo ℓ and p with $\psi_\ell(-1) = \psi_p(-1) = (-1)^k$ and let χ be the Dirichlet character modulo ℓp defined by $\chi(r) = \psi_\ell(r)\psi_p(r)$ for all $r \in \mathbb{Z}$. By abuse of notation, let ψ_p denote the "character" on the cosets $\mathcal{O}_K^\#/\mathcal{O}_K$ defined by setting $\psi_p((a \pm r/\sqrt{p})/2\lambda) = \psi_p(r)$ for any $a, r \in \mathbb{Z}$ satisfying $r \equiv ab \bmod 2\ell$. Then*

$$\varphi_{\psi_\ell}(\Theta F)(\tau) = \frac{1}{2\psi_\ell(b)} \cdot \Big(\varphi_{\overline{\psi_p}}(F)(4\ell\tau) \cdot \vartheta_\chi(\tau)\Big)\Big| U_p$$

where $\vartheta_\chi(\tau) = \sum_{r \in \mathbb{Z}} \chi(r) q^{r^2}$ is the twisted Jacobi theta series.

(ii) *Suppose $\ell = p$, and let ψ_p be a Dirichlet character mod p with $\psi_p(-1) = (-1)^k$. By abuse of notation, define ψ_p on $\mathcal{O}_K^\#/\mathcal{O}_K$ by $\psi_p(a/\lambda) = \psi_p(2a)$, $a \in \mathbb{Z}/p\mathbb{Z}$, where $\lambda = \varepsilon\sqrt{p}$ and ε is the fundamental unit of $\mathcal{O}_K$. Then*

$$\varphi_{\psi_p}(\Theta F)(\tau) = \Big(\varphi_{\psi_p}(F)(4p\tau) \cdot \vartheta(\tau)\Big)\Big| U_p,$$

where $\vartheta(\tau) = \sum_{r \in \mathbb{Z}} q^{r^2}$.

Proof.

(i) Write $f(\tau) = \varphi_{\overline{\psi_p}}(F)(\tau) = \sum_{\gamma \in \mathcal{O}_K^\#/\mathcal{O}_K} \sum_{n \in \mathbb{Z} + N_{K/\mathbb{Q}}(\gamma)} \overline{\psi_p(\gamma)} c(n, \gamma) q^{pn}$. In the product

$$\begin{aligned}f(4\ell\tau)\vartheta_\chi(\tau) &= \Big(\sum_{\gamma, n} \overline{\psi_p(\gamma)} c(n, \gamma) q^{4\ell pn}\Big)\Big(\sum_{r=-\infty}^\infty \chi(r) q^{r^2}\Big) \\ &= \sum_\gamma \sum_{n,r} \overline{\psi_p(\gamma)} \chi(r) c(n, \gamma) q^{4\ell pn + r^2},\end{aligned}$$

we get exponents which are divisible by p only when $n \in \mathbb{Z} + a^2/4\ell$ and $r \equiv \pm ab\,(2\ell)$ for some $a \in \mathbb{N}$ (which is uniquely determined mod 2ℓ). In this case $\gamma \in \gamma_{a,r} + \mathcal{O}_K$ with $\gamma_{a,r} = \frac{a \pm r/\sqrt{p}}{2\lambda}$ as before. Applying the U_p operator yields

$$\begin{aligned}\Big(f(4\ell\tau)\vartheta_\chi(\tau)\Big)\Big| U_p &= 2 \sum_{a \in \mathbb{Z}/2\ell\mathbb{Z}} \sum_{n \in \mathbb{Z} + a^2/4\ell} \sum_{r \equiv ab\,(2\ell)} \overline{\psi_p(\gamma_{a,r})} \chi(r) c(n - r^2/4p\ell, \gamma_{a,r}) q^{4\ell n} \\ &= 2 \sum_{a,n,r} \psi_\ell(ab) c(n - r^2/4p\ell, \gamma_{a,r}) q^{4\ell n} \\ &= 2\psi_\ell(b) \sum_{a,n} \psi_\ell(a) \tilde{c}(n, a) q^{4\ell n} \\ &= 2\psi_\ell(b) \varphi_{\psi_\ell}(\Theta F)(\tau).\end{aligned}$$

(ii) This is proved similarly to part (i). We do not need to divide by two, since the sum over r runs through only one congruence class (namely, $r \equiv ap\,(2p)$). The definition of ψ_p on $\mathcal{O}_K^{\#}/\mathcal{O}_K$ is such that $\psi_p(\gamma_a) = \psi_p(\frac{a(1+p)}{2\lambda}) = \psi_p(a)$ for all $a \in \mathbb{Z}/2p\mathbb{Z}$. $\square$

If we abbreviate

$$\Theta_\chi G = \Big(G(4\ell\tau)\cdot \vartheta_\chi(\tau)\Big)\Big| U_p$$

for a scalar valued modular form G, then the first item of the proposition (the case $\ell \neq p$) can be illustrated by the diagram

$$\begin{tikzcd} S_k(-N_{K/\mathbb{Q}}) \arrow[r, "\varphi_{\overline{\psi_p}}"] \arrow[d, "\Theta"] & S_k^{\epsilon,\mathrm{AL}}(p^2, \overline{\psi_p}\otimes\chi_p) \arrow[d, "\Theta_\chi"] \\ S_{k+1/2}((-2\ell)) \arrow[r, "\varphi_{\psi_\ell}"] & S_{k+1/2}^{\epsilon,\mathrm{AL}}(\ell^2, \psi_\ell) \end{tikzcd}$$

which commutes up to a constant factor. Note that the horizontal arrows are isomorphisms.

Example 12. Let $p = 5$. Fix the element $\lambda = 4 + \sqrt{5}$ of norm $\ell = 11$, and fix $b = 7$. We fix the Dirichlet characters ψ_{11} and ψ_5 by specifying $\psi_{11}(2) = e^{\pi i/5}$ and $\psi_5(2) = i$. Up to scalar multiples there is a unique cusp form of (antisymmetric) weight 5 for the dual Weil representation attached to $(\mathcal{O}_K, -N_{K/\mathbb{Q}})$ with $K = \mathbb{Q}(\sqrt{5})$, and it is

$$\begin{aligned} F(\tau) &= (q^{1/5} + 42q^{6/5} - 108q^{11/5}4q^{16/5} - 378q^{21/5} \pm \ldots)(\mathfrak{e}_{3/\sqrt{5}} - \mathfrak{e}_{2/\sqrt{5}}) \\ &\quad + (26q^{4/5} + 39q^{9/5} - 378q^{14/5} + 140q^{19/5} + 420q^{24/5} \pm \ldots)(\mathfrak{e}_{4/\sqrt{5}} - \mathfrak{e}_{1/\sqrt{5}}). \end{aligned}$$

One can compute F using, for example, the algorithm described in [8] (compare the example of Section 7 there); and after enough coefficients have been computed, one can identify its twisted component sum in $S_5(\Gamma_1(25))$ using standard methods for computing scalar-valued modular forms. The Doi-Naganuma lift of F is, up to a multiple, the well-known product s_5 of theta constants for $\mathbb{Q}(\sqrt{5})$ constructed by Gundlach ([5]; see also the example of Section 4 of [2]). The character ψ_5 on $\mathcal{O}_K^{\#}/\mathcal{O}_K$ is defined such that e.g.

$$\psi_5\left(1/\sqrt{5} + \mathcal{O}_K\right) = \psi_5\left(\frac{1 - 7/\sqrt{5}}{2\lambda} + \mathcal{O}_K\right) = \psi_5(7) = i.$$

Therefore the twisted component sum of F by $\overline{\psi_5}$ is the cusp form

$$\varphi_{\overline{\psi_5}}(F)(\tau) = 2q + 52iq^4 + 84q^6 + 78iq^9 - 216q^{11} - 756iq^{14} - 8q^{16} \pm \ldots \in S_5(\Gamma_0(25), \overline{\psi_5}\otimes\chi_5) = S_5(\Gamma_0(25), \psi_5).$$

After multiplying

$$\begin{aligned} \varphi_{\overline{\psi_5}}(F)(44\tau)\vartheta_\chi(\tau) &= \Big(2q^{44} + 52iq^{176} \pm \ldots\Big)\Big(2q + 2\zeta_{20}^7 q^4 - 2\zeta_{20}^9 q^9 \pm \ldots\Big) \\ &= 4q^{45} + 4\zeta_{20}^7 q^{48} - 4\zeta_{20} q^{53} - 4\zeta_{20}^4 q^{60} \pm \ldots \end{aligned}$$

and applying U_5 we get the series

$$4q^9 - 4\zeta_{20}^4 q^{12} + 4\zeta_{20}^{18} q^{16} + 4\zeta_{20}^2 q^{25} - 104\zeta_{20}^2 q^{36} \pm \ldots$$

Dividing by $2\psi_{11}(b) = -2\zeta_{20}^4$ yields the twisted component sum of the theta contraction ΘF:

$$\varphi_{\psi_{11}}(\Theta F)(\tau) = 2\zeta_{20}^6 q^9 + 2q^{12} + 2\zeta_{20}^4 q^{16} - 2\zeta_{20}^{18} q^{25} + 52\zeta_{20}^{18} q^{36} \pm \ldots$$

From this we can read off the Shimura lift of the underlying vector-valued modular form ΘF: the coefficient of q^n is zero if $11|n$, and otherwise $\sum_{d|n} \frac{1}{2\psi_{11}(d)}(n/d)^{5-1}c(d^2)$ if $c(n)$ is the coefficient of q^n in $\varphi_{\psi_{11}}(\Theta F)(\tau)$, so

$$s_5(\lambda\tau, \lambda'\tau) = -q^3 + q^4 + q^5 + 10q^6 - 10q^8 - 121q^9 + 98q^{10} + 275q^{12} + 32q^{13} + 140q^{14} \pm \ldots \in S_{10}(\Gamma_0(11)).$$

Example 13. Let $p = 13$. Fix the totally positive element $\lambda = \frac{13+3\sqrt{13}}{2}$ of norm $\ell = 13$ and fix $b = 13$. We fix an odd Dirichlet character ψ_{13} mod 13 by specifying $\psi_{13}(2) = \zeta_{12} = e^{\pi i/6}$. The dual Weil representation attached to $(\mathcal{O}_K, -N_{K/\mathbb{Q}})$, $K = \mathbb{Q}(\sqrt{13})$ admits up to scalar multiples a unique cusp form of weight 3:

$$\begin{aligned} F(\tau) = &(q^{1/13} - 33q^{14/13} + 27q^{27/13} + 33q^{40/13} \pm ...)(\mathfrak{e}_{1/\lambda} - \mathfrak{e}_{12/\lambda}) \\ &+ (3q^{3/13} + 5q^{16/13} + 42q^{29/13} - 99q^{42/13} \pm ...)(\mathfrak{e}_{4/\lambda} - \mathfrak{e}_{9/\lambda}) \\ &+ (-7q^{4/13} - 3q^{17/13} - 33q^{30/13} + 49q^{43/13} \pm ...)(\mathfrak{e}_{2/\lambda} - \mathfrak{e}_{11/\lambda}) \\ &+ (0q^{9/13} - 22q^{22/13} + 33q^{35/13} + 15q^{48/13} \pm ...)(\mathfrak{e}_{3/\lambda} - \mathfrak{e}_{10/\lambda}) \\ &+ (11q^{10/13} - 12q^{23/13} + 0q^{36/13} + 50q^{49/13} \pm ...)(\mathfrak{e}_{6/\lambda} - \mathfrak{e}_{7/\lambda}) \\ &+ (21q^{12/13} + 14q^{25/13} - 66q^{38/13} + 9q^{51/13} \pm ...)(\mathfrak{e}_{5/\lambda} - \mathfrak{e}_{8/\lambda}). \end{aligned}$$

Under the Doi-Naganuma lift, F is mapped to the cusp form ω_3 used by van der Geer and Zagier to compute the ring of Hilbert modular forms for $\mathcal{O}_K$ ([4], Section 10). The twisted component sum of F by ψ_{13} is

$$\varphi_{\psi_{13}}(F)(\tau) = 2\zeta_{12}q + 6\zeta_{12}^3q^3 - 14\zeta_{12}^2q^4 - 22q^{10} - 42\zeta_{12}^4q^{12} \pm ... \in S_3(\Gamma_0(169), \psi_{13} \otimes \chi_{13}).$$

With this we can compute $\omega_3(\lambda\tau, \lambda'\tau)$ as follows: multiply

$$\varphi_{\psi_{13}}(F)(52\tau) \cdot \vartheta(\tau) = 2\zeta_{12}q^{52} + 4\zeta_{12}q^{53} + 4\zeta_{12}q^{56} \pm ...$$

and apply the Hecke operator U_{13} to obtain

$$\varphi_{\psi_{13}}(\Theta F)(\tau) = 2\zeta_{12}q^4 + 6\zeta_{12}^3q^{12} - 14\zeta_{12}^2q^{16} + 4\zeta_{12}q^{17} + 12\zeta_{12}^3q^{25} \pm ...$$

and therefore the Shimura lift

$$\omega_3(\lambda\tau, \lambda'\tau) = q^2 - 3q^4 - 6q^5 + 9q^6 - q^8 + 6q^9 + 57q^{10} \pm ... \in S_6(\Gamma_0(13)).$$

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