VECTOR-VALUED EISENSTEIN SERIES OF SMALL WEIGHT

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Abstract. We study the (mock) Eisenstein series E_k of weight $k \in \{1, 3/2, 2\}$ for the Weil representation on an even lattice, defined as the result of Bruinier and Kuss's coefficient formula for the Eisenstein series naively evaluated at k. We describe the transformation law of E_k in general. Most of this note is dedicated to collecting examples where the coefficients of E_k contain interesting arithmetic information. Finally we make a few remarks about the case $k = 1/2$.

Key words and phrases: Modular forms; Mock modular forms; Eisenstein series; Weil representation

1. Introduction

In [5], Bruinier and Kuss give an expression for the Fourier coefficients of the Eisenstein series E_k of weight $k \geq 5/2$ for the Weil representation attached to a discriminant form. These coefficients involve special values of L-functions and zero counts of polynomials modulo prime powers, and they also make sense for $k \in \{1, 3/2, 2\}$. Unfortunately, the q-series E_k obtained in this way often fail to be modular forms. In particular, in weight $k = 3/2$ and $k = 2$, the Eisenstein series may be a mock modular form that requires a real-analytic correction in order to transform as a modular form. Many examples of this phenomenon of the Eisenstein series are well-known (although perhaps less familiar in a vector-valued setting). We will list a few examples of this:

Example 1. The Eisenstein series of weight 2 for a unimodular lattice Λ is the quasimodular form

$$E_2(\tau) = 1 - 24\sum_{n=1}^{\infty} \sigma_1(n) q^n = 1 - 24q - 72q^2 - 96q^3 - 168q^4 - ...$$

where $\sigma_1(n) = \sum_{d|n} d$, which transforms under the modular group by

$$E_2\Big(\frac{a\tau + b}{c\tau + d}\Big) = (c\tau + d)^2 E_2(\tau) + \frac{6}{\pi i} c(c\tau + d).$$

Here and in the examples below, ρ^* denotes a Weil representation and $\mathfrak{e}_\gamma$ are basis vectors in the underlying space of ρ^*. (These are defined in section 2.)

Example 2. The Eisenstein series of weight 3/2 for the quadratic form $q_2(x) = x^2$ is essentially Zagier's mock Eisenstein series:

$$E_{3/2}(\tau) = \Big(1 - 6q - 12q^2 - 16q^3 - ...\Big)\mathfrak{e}_0 + \Big(- 4q^{3/4} - 12q^{7/4} - 12q^{11/4} - ...\Big)\mathfrak{e}_{1/2},$$

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in which the coefficient of $q^{n/4}\mathfrak{e}_{n/2}$ is -12 times the Hurwitz class number $H(n)$. It transforms under the modular group by

$$E_{3/2}\Big(\frac{a\tau+b}{c\tau+d}\Big) = (c\tau+d)^{3/2}\rho^*\Big(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\Big)\Big[E_{3/2}(\tau) - \frac{3}{\pi}\sqrt{\frac{i}{2}}\int_{d/c}^{i\infty}(\tau+t)^{-3/2}\vartheta(t)\,\mathrm{d}t\Big],$$

where ϑ is the theta series

$$\vartheta(\tau) = \sum_{n\in\mathbb{Z}} q^{n^2/4}\mathfrak{e}_{n/2}.$$

Example 3. In the Eisenstein series of weight $3/2$ for the quadratic form $q_3(x) = 6x^2$, the components of $\mathfrak{e}_{1/12}$, $\mathfrak{e}_{5/12}$, $\mathfrak{e}_{7/12}$ and $\mathfrak{e}_{11/12}$ are

$$\Big(-3q^{23/24} - 5q^{47/24} - 7q^{71/24} - 8q^{95/24} - 10q^{119/24} - 10q^{143/24} - ...\Big)\mathfrak{e}_\gamma$$

for $\gamma \in \{1/12, 5/12, 7/12, 11/12\}$. We verified by computer that the coefficient of $q^{n-1/24}$ above is (-1) times the degree of the n-th partition class polynomial considered by Bruinier and Ono [6] for $1 \le n \le 750$, which is not surprising in view of example 2 since this degree also counts equivalence classes of certain binary quadratic forms. This Eisenstein series is not a modular form.

Example 4. The Eisenstein series of weight $3/2$ for the quadratic form $q_4(x,y,z) = x^2+y^2-z^2$ is a mock modular form that is related to the functions considered by Bringmann and Lovejoy [1] in their work on overpartitions. More specifically, the component of $\mathfrak{e}_{(0,0,0)}$ in $E_{3/2}$ is

$$1 - 2q - 4q^2 - 8q^3 - 10q^4 - ... = 1 - \sum_{n=1}^\infty |\overline{\alpha}(n)|q^n,$$

where $\overline{\alpha}(n)$ is the difference between the number of even-rank and odd-rank overpartitions of n. Similarly, the $M2$-rank differences considered in [1] appear to occur in the Eisenstein series of weight $3/2$ for the quadratic form $q_5(x,y,z) = 2x^2+2y^2-z^2$, whose $\mathfrak{e}_{(0,0,0)}$-component is

$$1 - 2q - 4q^2 - 2q^4 - 8q^5 - 8q^6 - 8q^7 - ...$$

Example 5. Unlike the previous examples, the Eisenstein series of weight $3/2$ for the quadratic form $q_6(x,y,z) = -x^2-y^2-z^2$ is a true modular form; in fact, it is the theta series for the cubic lattice and the Fourier coefficients of its $\mathfrak{e}_{(0,0,0)}$-component count the representations of integers as sums of three squares.

Among negative-definite lattices of small dimension there are lots of examples where the Eisenstein series equals the theta series. (Note that we find theta series for negative-definite lattices instead of positive-definite because we consider the dual Weil representation ρ^*.) When the lattice is even-dimensional this immediately leads to formulas for representation numbers in terms of twisted divisor sums. These formulas are of course well-known but the vector-valued derivations of these formulas seem more natural than the usual derivation as identities among scalar-valued forms of higher level. We give several examples of this throughout the note.

In the last section we make some remarks about the case $k = 1/2$, where the formula of [5] no longer makes sense and so the methods of this note break down.

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2. Background

In this section we review some facts about the metaplectic group and vector-valued modular forms, as well as Dirichlet L-functions, which will be useful later.

Recall that the **metaplectic group** $Mp_2(\mathbb{Z})$ is the double cover of $SL_2(\mathbb{Z})$ consisting of pairs (M, ϕ), where $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, and ϕ is a branch of $\sqrt{c\tau + d}$ on the upper half-plane

$$\mathbb{H} = \{\tau = x + iy \in \mathbb{C} : \, y > 0\}.$$

We will usually omit ϕ. $Mp_2(\mathbb{Z})$ is generated by the elements

$$T = \Big(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1 \Big), \;\; S = \Big(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau} \Big),$$

with defining relations $S^8 = I$ and $S^2 = (ST)^3$.

Let Λ be a lattice (which we can always take as $\Lambda = \mathbb{Z}^e$ for some $e \in \mathbb{N}$) with an even quadratic form $q : \Lambda \to \mathbb{Z}$, and let

$$\Lambda' = \{v \in \Lambda \otimes \mathbb{Q}^e : \, \langle v, w \rangle \in \mathbb{Z} \, \text{for all} \, w \in \Lambda\}$$

be the dual lattice. We denote by $\mathfrak{e}_\gamma$, $\gamma \in \Lambda'/\Lambda$ the natural basis of the group algebra $\mathbb{C}[\Lambda'/\Lambda]$. The **Weil representation** of $Mp_2(\mathbb{Z})$ attached to Λ is the map

$$\rho : Mp_2(\mathbb{Z}) \longrightarrow \mathrm{Aut}\, \mathbb{C}[\Lambda'/\Lambda]$$

defined by

$$\rho(T)\mathfrak{e}_\gamma = \mathbf{e}\Big(q(\gamma)\Big)\mathfrak{e}_\gamma, \;\; \rho(S)\mathfrak{e}_\gamma = \frac{\sqrt{i}^{b^- - b^+}}{\sqrt{|\Lambda'/\Lambda|}} \sum_{\beta \in \Lambda'/\Lambda} \mathbf{e}\Big(- \langle \gamma, \beta \rangle \Big) \mathfrak{e}_\beta.$$

In particular,

$$\rho(Z)\mathfrak{e}_\gamma = i^{b^- - b^+} \mathfrak{e}_{-\gamma}, \;\; \text{where} \;\; Z = (-I, i) = S^2 = (ST)^3.$$

Here we use $\mathbf{e}(x)$ to denote $e^{2\pi i x}$, and (b^+, b^-) is the signature of Λ.
We will usually consider the dual representation ρ^* of ρ (which also occurs as the Weil representation itself, for the lattice Λ and quadratic form $-q$).

A **modular form of weight** k for ρ^* is a holomorphic function $f : \mathbb{H} \to \mathbb{C}[\Lambda'/\Lambda]$ with the properties:
(i) f transforms under the action of $Mp_2(\mathbb{Z})$ by

$$f(M \cdot \tau) = (c\tau + d)^k \rho^*(M) f(\tau), \;\; M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Mp_2(\mathbb{Z}),$$

where the branch of $(c\tau+d)^k$ is prescribed by M as an element of $Mp_2(\mathbb{Z})$ if k is half-integer;
(ii) f is holomorphic in ∞. This means that in the Fourier expansion

$$f(\tau) = \sum_{\gamma \in \Lambda'/\Lambda} \sum_{n \in \mathbb{Z} - q(\gamma)} c(n,\gamma) q^n \mathfrak{e}_\gamma,$$

all coefficients $c(n,\gamma)$ are zero for $n < 0$.

If N is the smallest natural number such that $N\langle \gamma, \beta \rangle$ and $Nq(\gamma) \in \mathbb{Z}$ for all $\beta, \gamma \in \Lambda'/\Lambda$, then ρ^* factors through $SL_2(\mathbb{Z}/N\mathbb{Z})$ if $e = \dim \Lambda$ is even, and through a double cover of $SL_2(\mathbb{Z}/N\mathbb{Z})$ if e is odd. This implies in particular that the component functions f_γ of f are scalar modular forms of level N.

When studying the weight $3/2$ Eisenstein series we will consider **harmonic weak Maass forms**, which have the same transformation behavior as modular forms but for which the holomorphy assumption is weakened to real-analyticity and the weight-k Laplace equation $\Delta f(\tau) = 2iky \frac{\partial}{\partial \overline{\tau}} f(\tau)$, where $\Delta = y^2 (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$ is the hyperbolic Laplacian on $\mathbb{H}$. Harmonic weak Maass forms are also required to satisfy a growth condition of the form $|f(\tau)| < Ce^{Ny}$ at ∞. We refer to [3] and [12] for details.

The weights of modular forms are restricted due to

$$f(\tau) = f(Z \cdot \tau) = i^{2k} \rho^*(Z) f(\tau) = i^{2k+b^+-b^-} \sum_{\gamma \in \Lambda'/\Lambda} f_{-\gamma}(\tau) \mathfrak{e}_\gamma.$$

In particular, if $2k + b^+ - b^-$ is not an even integer, then there are no nonzero modular forms. In the case $2k + b^+ - b^- \equiv 2\,(4)$ (which seems to be of less interest), the components satisfy $f_\gamma = -f_{-\gamma}$ and in particular the $\mathfrak{e}_0$-component of f must be zero. We will consider only the case $2k + b^+ - b^- \equiv 0\,(4)$ as we are interested in Eisenstein series with constant term $1 \cdot \mathfrak{e}_0$.

Remark 6. There is an involution $\sim$ of the metaplectic group given on the standard generators by

$$\tilde{S} = S^{-1}, \;\; \tilde{T} = T^{-1},$$

which is well-defined because

$$(\tilde{S}\tilde{T})^3 = S^{-1}(ST)^{-3}S = S^{-1}S^{-2}S = \tilde{S}^2$$

and $\tilde{S}^8 = I$. On matrices it is given by

$$\widetilde{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix},$$

and it acts on the branches of square roots by $\tilde{\phi}(\tau) = \overline{\phi(-\overline{\tau})}$, where $\phi(\tau)^2 = c\tau + d$. One can check on the generators S, T that this intertwines the Weil representation ρ and its dual ρ^* in the sense that

$$\rho(\tilde{M}) = \rho^*(M) = \overline{\rho(M)}, \;\; M \in Mp_2(\mathbb{Z}).$$

Remark 7. At many points in this note we will need to consider the L-function

$$L(s,\chi_D) = \sum_{n=1}^{\infty} \chi_D(n) n^{-s}$$

attached to the Dirichlet character mod $|D|$,

$$\chi_D(n) = \Big(\frac{D}{n}\Big),$$

where D is a discriminant (i.e. $D \equiv 0, 1 \bmod 4$). In particular, we recall the following properties of Dirichlet L-functions.
(i) Let χ be a Dirichlet character. Then $L(s,\chi)$ converges absolutely in some half-plane $\mathrm{Re}[s] > s_0$ and is given by an Euler product

$$L(s,\chi) = \prod_{p \text{ prime}} (1 - \chi(p)p^{-s})^{-1}$$

there.
(ii) $L(s,\chi)$ has a meromorphic extension to all $\mathbb{C}$ and satisfies the functional equation

$$\Gamma(s)\cos\Big(\frac{\pi(s-\delta)}{2}\Big) L(s,\chi) = \frac{\tau(\chi)}{2i^{\delta}} (2\pi/f)^s L(1-s,\overline{\chi}),$$

where f is the conductor of χ, $\tau(\chi) = \sum_{a=1}^{f} \chi(a) e^{2\pi i a/f}$ is the Gauss sum of χ, and

$$\delta = \begin{cases} 1: & \chi(-1) = -1; \\ 0: & \chi(-1) = 1. \end{cases}$$

(iii) $L(s,\chi)$ is never zero at $s = 1$, and is holomorphic there unless χ is a trivial character, in which case it has a simple pole.
(iv) The values $L(1-n,\chi)$, $n \in \mathbb{N}$ are rational numbers, given by

$$L(1-n,\chi) = -\frac{B_{n,\chi}}{n},$$

where $B_{n,\chi} \in \mathbb{Q}$ is a generalized Bernoulli number.
We refer to section 4 of [15] for these and other results on Dirichlet L-functions.

3. The real-analytic Eisenstein series

Fix an even lattice Λ and let ρ^* be the dual Weil representation on $\mathbb{C}[\Lambda'/\Lambda]$.

Definition 8. The **real-analytic Eisenstein series** of weight k is

$$E_k^*(\tau,s) = \sum_{M \in \tilde{\Gamma}_\infty \backslash \Gamma} (y^s \mathfrak{e}_0)|_k M = \frac{y^s}{2} \sum_{c,d} (c\tau+d)^{-k} |c\tau+d|^{-2s} \rho^*(M)^{-1} \mathfrak{e}_0.$$

Here, (c,d) runs through all pairs of coprime integers and M is any element $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Mp_2(\mathbb{Z})$ with bottom row (c,d); and the branch of $(c\tau+d)^{-k}$ is determined by M as an element of $Mp_2(\mathbb{Z})$ as usual.

This series converges absolutely and locally uniformly in the half-plane $\mathrm{Re}[s] > 1 - k/2$ and defines a holomorphic function in s. For fixed s, it transforms under the metaplectic group by

$$E_k^*\Big(M \cdot \tau, s\Big) = (c\tau + d)^k \rho^*(M) E_k^*(\tau, s)$$

for any $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Mp_2(\mathbb{Z})$. These series were considered by Bruinier and Kühn [4] in weight $k \geq 2$ who also give expressions for their Fourier expansions. (More generally they consider the series obtained after replacing $\mathfrak{e}_0$ with $\mathfrak{e}_\beta$ for an element $\beta \in \Lambda'/\Lambda$ with $q(\beta) \in \mathbb{Z}$. We do not do this because it seems to make the formulas below considerably more complicated, and because for many discriminant forms Λ'/Λ one can obtain the real-analytic Eisenstein series associated to any β from the $E_k^*(\tau, s)$ above by a simple "averaging" argument. See for example the appendix of [16].)

The series $E_k^*(\tau, s)$ can be analytically extended beyond the half-plane $\mathrm{Re}[s] > 1 - k/2$. We will focus here on weights $k \in \{1, 3/2, 2\}$, in which the Fourier series is enough to give an explicit analytic continuation to $s = 0$. First we work out an expression for the Fourier series (in particular, our result below differs in appearance from [4] because we use a different computation of the Euler factors). Writing

$$E_k^*(\tau, s) = \mathfrak{e}_0 + \sum_{\gamma \in \Lambda'/\Lambda} \sum_{n \in \mathbb{Z} - q(\gamma)} c(n, \gamma, s, y) q^n \mathfrak{e}_\gamma,$$

a computation analogous to section 1.2.3 of [2] using the exact formula for the coefficients $\rho(M)_{0,\gamma}$ of the Weil representation cited there shows that

$$\begin{aligned} c(n, \gamma, s, y) &= \frac{y^s}{2} \sum_{c \neq 0} \sum_{d \in (\mathbb{Z}/c\mathbb{Z})^\times} \rho(M)_{0,\gamma} \int_{-\infty + iy}^{\infty + iy} (c\tau + d)^{-k} |c\tau + d|^{-2s} \mathbf{e}(-n\tau)\, \mathrm{d}x \\ &= y^s \sum_{c=1}^\infty \sum_{d \in (\mathbb{Z}/c\mathbb{Z})^\times} \rho(M)_{0,\gamma} c^{-k-2s} \mathbf{e}\Big(\frac{nd}{c}\Big) \int_{-\infty+iy}^{\infty+iy} \tau^{-k} |\tau|^{-2s} \mathbf{e}(-n\tau)\, \mathrm{d}x \\ &= \frac{\sqrt{i}^{b^- - b^+}}{\sqrt{|\Lambda'/\Lambda|}} \tilde{L}(n, \gamma, k + e/2 + 2s) I(k, y, n, s), \end{aligned} \tag{1}$$

where M is any element of $Mp_2(\mathbb{Z})$ whose bottom row is (c, d). Here, $\tilde{L}(n, \gamma, s)$ is the L-series

$$\begin{aligned}\tilde{L}(n,\gamma,s) &= \sum_{c=1}^{\infty} c^{-s+e/2} \sum_{d\in(\mathbb{Z}/c\mathbb{Z})^\times} \rho(M)_{0,\gamma}\mathbf{e}\Big(\frac{nd}{c}\Big) \\ &= \sum_{c=1}^{\infty} c^{-s} \sum_{\substack{v\in\Lambda/c\Lambda \\ d\in(\mathbb{Z}/c\mathbb{Z})^\times}} \mathbf{e}\Big(\frac{aq(v)-\langle\gamma,v\rangle+dq(\gamma)-nd}{c}\Big) \\ &= \sum_{c=1}^{\infty} c^{-s} \sum_{a|c} \Big[\mu(c/a)a(c/a)^e \cdot \#\Big\{v\in\Lambda/a\Lambda :\ q(v-\gamma)+n\equiv 0 \,(\mathrm{mod}\, a)\Big\}\Big] \\ &= \zeta(s-e)^{-1}L(n,\gamma,s-1),\end{aligned}$$

where $L(n,\gamma,s)$ is

$$L(n,\gamma,s) = \sum_{a=1}^{\infty} a^{-s}\mathbf{N}(a) = \prod_{p\text{ prime}} \Big(\sum_{\nu=0}^{\infty} p^{-\nu s}\mathbf{N}(p^\nu)\Big) = \prod_{p\text{ prime}} L_p(n,\gamma,s),$$

and $\mathbf{N}(p^\nu)$ is the number of zeros $v\in\Lambda/p^\nu\Lambda$ of the quadratic polynomial $q(v-\gamma)+n$; and $I(k,y,n,s)$ is the integral

$$\begin{aligned}I(k,y,n,s) &= y^s\int_{-\infty+iy}^{\infty+iy} \tau^{-k}|\tau|^{-2s}\mathbf{e}(-n\tau)\,\mathrm{d}x \\ &= y^{1-k-s}e^{2\pi ny}\int_{-\infty}^{\infty}(t+i)^{-k}(t^2+1)^{-s}\mathbf{e}(-nyt)\,\mathrm{d}t, \quad \tau=y(t+i).\end{aligned}$$

Remark 9. Both the L-series term $\tilde{L}(n,\gamma,s)$ and the integral term $I(k,y,n,s)$ of (1) have meromorphic continuations to all $s\in\mathbb{C}$. First we remark that the integral $I(k,y,n,s)$ was considered by Gross and Zagier [10], section IV.3., where it was shown that for $n\neq 0$, $I(k,y,n,s)$ is a finite linear combination of K-Bessel functions (we will not need the exact expression) and its value at $s=0$ is given by

$$(2)\qquad I(k,y,n,0) = \begin{cases} 0: & n<0; \\ (-2\pi i)^k n^{k-1}\frac{1}{\Gamma(k)}: & n>0; \end{cases}$$

if $n\neq 0$; and when $n=0$,

$$(3)\qquad I(k,y,0,s) = \pi(-i)^k 2^{2-k-2s}y^{1-k-s}\frac{\Gamma(2s+k-1)}{\Gamma(s)\Gamma(s+k)}.$$

In particular, the zero value of the latter expression is

$$I(k,y,0,0) = \begin{cases} 0: & k\neq 1; \\ -i\pi: & k=1. \end{cases}$$

The Euler factors $L_p(n,\gamma,s)=\sum_{\nu=0}^{\infty}p^{-\nu s}\mathbf{N}(p^\nu)$ are known to be rational functions in p^{-s} that can be calculated using the methods of [7] (see also section 6 of [16] as well as the appendix, where the result of the case $p=2$ was worked out). For generic primes (primes $p\neq 2$ that do not divide $|\Lambda'/\Lambda|$, or the numerator or denominator of n if $n \neq 0$) the result is that

$$L_p(n,\gamma,s) = \begin{cases} \frac{1}{1-p^{e-1-s}}\Big[1 - \Big(\frac{D'}{p}\Big)p^{e/2-s}\Big] : & n \neq 0; \\ \frac{1-\Big(\frac{D'}{p}\Big)p^{e/2-1-s}}{(1-p^{e-1-s})\Big[1-\Big(\frac{D'}{p}\Big)p^{e/2-s}\Big]} : & n = 0; \end{cases}$$

if e is even and

$$L_p(n,\gamma,s) = \begin{cases} \frac{1}{1-p^{e-1-s}}\Big[1 + \Big(\frac{\mathcal{D}'}{p}\Big)p^{(e-1)/2-s}\Big] : & n \neq 0; \\ \frac{1-p^{e-1-2s}}{(1-p^{e-1-s})(1-p^{e-2s})} : & n = 0; \end{cases}$$

if e is odd. Here, D' and $\mathcal{D}'$ are defined by

$$D' = (-1)^k |\Lambda'/\Lambda| \text{ and } \mathcal{D}' = 2nd_\gamma^2(-1)^{k-1/2}|\Lambda'/\Lambda|,$$

where $d_\gamma \in \mathbb{N}$ is minimal such that $d_\gamma \gamma \in \Lambda$.

In particular, if we define $D = D' \cdot \prod_{\text{bad } p} p^2$ and $\mathcal{D} = \mathcal{D}' \cdot \prod_{\text{bad } p} p^2$, where the bad primes are 2 and any prime dividing $|\Lambda'/\Lambda|$ or n, then we get the meromorphic continuations

$$\tilde{L}(n,\gamma,s) = \begin{cases} \frac{1}{L(s-e/2,\chi_D)} \prod_{\text{bad } p}(1-p^{e-s})L_p(n,\gamma,s-1) : & n \neq 0; \\ \frac{L(s-1-e/2,\chi_D)}{L(s-e/2,\chi_D)} \prod_{\text{bad } p}(1-p^{e-s})L_p(s-1) : & n = 0; \end{cases}$$

if e is even and

$$\tilde{L}(n,\gamma,s) = \begin{cases} \frac{L(s-(e+1)/2,\chi_{\mathcal{D}})}{\zeta(2s-1-e)} \prod_{\text{bad } p} \frac{1-p^{e-s}}{1-p^{e+1-2s}} L_p(n,\gamma,s-1) : & n \neq 0; \\ \frac{\zeta(2s-2-e)}{\zeta(2s-1-e)} \prod_{\text{bad } p} \frac{(1-p^{e-s})(1-p^{e+2-2s})}{1-p^{e+1-2s}} L_p(s-1) : & n = 0; \end{cases}$$

if e is odd.

Remark 10. We denote by E_k the series

$$E_k(\tau) = \mathfrak{e}_0 + \sum_{\gamma \in \Lambda'/\Lambda} \sum_{n>0} c(n,\gamma,0,y) q^n \mathfrak{e}_\gamma.$$

The formula (2) gives $I(k,y,n,0) = (-2\pi i)^k n^{k-1} \frac{1}{\Gamma(k)}$ independently of y, and so $E_k(\tau)$ is holomorphic. When $k > 2$, this is just the zero-value $E_k(\tau) = E_k^*(\tau,0)$ and therefore E_k is a modular form. In small weights this tends to fail because the terms

$$\lim_{s \to 0} \tilde{L}(n,\gamma,k+e/2+2s) I(k,y,n,s)$$

may have a pole of $\tilde{L}$ cancelling the zero of I for $n \leq 0$, resulting in nonzero (and often nonholomorphic) contributions to $E_k^*(\tau,0)$.

Remark 11. Suppose the dimension e is even; then we can apply theorem 4.8 of [5] to get a simpler coefficient formula. (The condition $k = e/2$ there is only necessary for their computation of local L-factors, which we do not use.) It follows that the coefficient $c(n,0)$ of $q^n\mathfrak{e}_0$ in E_k is

$$c(n,0) = \frac{(2\pi)^k(-1)^{(2k+b^+-b^-)/4}}{L(k,\chi_D)\sqrt{|\Lambda'/\Lambda|}\Gamma(k)}\cdot\sigma_{k-1}(n,\chi_D)\cdot\prod_{p|D'}\Big[(1-p^{e/2-k})L_p(n,0,k+e/2-1)\Big],$$

where $\sigma_{k-1}(n,\chi_D)$ is the twisted divisor sum

$$\sigma_{k-1}(n,\chi_D) = \sum_{d|n}\chi_D(n/d)d^{k-1}$$

and $D' = 4|\Lambda'/\Lambda|$. For a fixed lattice Λ, the expression $\prod_{p|D'}\Big[(1-p^{e/2-k})L_p(n,0,k+e/2-1)\Big]$ can always be worked out exactly using the method of [7], although this can be somewhat tedious (in particular the case $p=2$, which was worked out explicitly in the appendix of [16].) A worksheet in SAGE to compute these expressions is available on the author's university webpage, and was used to compute the examples in the following sections. Theorem 4.8 of [5] also gives an interpretation of the coefficients when e is odd but this is more complicated.

4. Weight 1

In weight 1, the L-series term is always holomorphic at $s=0$. However, the zero-value

$$I(1,y,0,0) = -i\pi$$

being nonzero means that E_k still needs a correction term. Setting $s=0$ in the real-analytic Eisenstein series gives

$$\begin{aligned} E_1^*(\tau,0) = E_1(\tau) - \pi\frac{(-1)^{(2+b^--b^+)/4}}{\sqrt{|\Lambda'/\Lambda|}}\frac{L(0,\chi_D)}{L(1,\chi_D)} \\ \cdot\sum_{\substack{\gamma\in\Lambda'/\Lambda \\ q(\gamma)\in\mathbb{Z}}}\Big[\prod_{\text{bad } p}\lim_{s\to 0}(1-p^{e/2-1-2s})L_p(0,\gamma,e/2+2s)\Big]\mathfrak{e}_\gamma,\end{aligned}$$

where D is the discriminant $D = -4|\Lambda'/\Lambda|$ and the bad primes are the primes dividing D. In particular, E_1 may differ from the true modular form $E_1^*(\tau,0)$ by a constant. (Of course, $E_1^*(\tau,0)$ may be identically zero.)

For two-dimensional negative-definite lattices, the corrected Eisenstein series $E_1^*(\tau,0)$ is often a multiple of the theta series. This leads to identities relating representation numbers of quadratic forms and divisor counts. Of course, such identities are well-known from the theory of modular forms of higher level. The vector-valued proofs tend to be shorter since $M_k(\rho^*)$ is generally much smaller than the space of modular forms of higher-level in which the individual components lie, so there is less algebra (although computing the local factors takes some work). We give two examples here.

Example 12. Consider the quadratic form $q(x,y) = -x^2 - xy - y^2$, with $|\Lambda'/\Lambda| = 3$. The L-function values are

$$L(0,\chi_{-12}) = \frac{2}{3}, \;\; L(1,\chi_{-12}) = \frac{\pi\sqrt{3}}{6}$$

and the local L-series are

$$L_2(0,0,s) = \frac{1+2^{-s}}{1-2^{2-2s}}, \;\; L_3(0,0,s) = \frac{1}{1-3^{1-s}}$$

with

$$\lim_{s\to 0}(1-2^{-2s})L_2(0,0,1+2s) = \frac{3}{4}, \;\; \lim_{s\to 0}(1-3^{-2s})L_3(0,0,1+2s) = 1,$$

and therefore $E_1^*(\tau,0) = E_1(\tau) + \mathfrak{e}_0$. Since $M_1(\rho^*)$ is one-dimensional (which one can compute by identifying $\Delta \cdot M_1(\rho^*) \subseteq M_{13}(\rho^*)$ using [16], for example, where Δ is the discriminant), comparing constant terms shows that

$$E_1(\tau) + \mathfrak{e}_0 = 2\vartheta.$$

Using remark 11, we find that the coefficient $c(n,0)$ of $q^n \mathfrak{e}_0$ in E_1 is

$$\begin{aligned} c(n,0) &= \frac{2\pi}{L(1,\chi_{-12})\cdot\sqrt{3}} \cdot \sigma_0(n,\chi_{-12}) \\ &\quad \cdot \underbrace{\begin{cases} 3/2: & v_2(n)\text{ even;} \\ 0: & v_2(n)\text{ odd;} \end{cases}}_{\text{local factor at 2}} \cdot \underbrace{\begin{cases} 2: & n \neq (3a+2)3^b \text{ for any } a,b\in\mathbb{N}_0; \\ 0: & n = (3a+2)3^b \text{ for some } a,b\in\mathbb{N}_0; \end{cases}}_{\text{local factor at 3}} \\ &= 12\Big[\sum_{d|n}\Big(\frac{-12}{d}\Big)\Big] \cdot \begin{cases} 1: & n \neq (3a+2)3^b; \\ 0: & n = (3a+2)3^b. \end{cases} \end{aligned}$$

This implies the identity

$$\begin{aligned} &\#\{(a,b)\in\mathbb{Z}^2 : \, a^2+ab+b^2 = n\} \\ &= 6\varepsilon \cdot \Big(\#\{\text{divisors}\, d = 6\ell+1 \,\text{of}\, n\} - \#\{\text{divisors}\, d = 6\ell-1 \,\text{of}\, n\}\Big), \end{aligned}$$

valid for $n \geq 1$, where $\varepsilon = 1$ unless n has the form $(3a+2)3^b$ for $a,b\in\mathbb{N}_0$, in which case $\varepsilon = 0$.

Example 13. Consider the quadratic form $q(x,y) = -x^2 - y^2$, with $|\Lambda'/\Lambda| = 4$ and $\chi_{-16} = \chi_{-4}$. The L-function values are

$$L(0,\chi_{-4}) = \frac{1}{2}, \;\; L(1,\chi_{-4}) = \frac{\pi}{4},$$

and the only bad prime is 2 with $L_2(0,0,s) = \frac{1}{1-2^{1-s}}$ and therefore

$$\lim_{s\to 0}(1-2^{e/2-1-2s})L_2(0,0,e/2+2s) = 1.$$

Therefore,

$$E_1^*(\tau,0) = E_1(\tau) + \mathfrak{e}_0.$$

Since $M_1(\rho^*)$ is one-dimensional, comparing constant terms gives $E_1(\tau) + \mathfrak{e}_0 = 2\vartheta(\tau)$.

By remark 11, the coefficient $c(n,0)$ of $q^n \mathfrak{e}_0$ in E_1 is

$$c(n,0) = \frac{2\pi}{L(1,\chi_{-4}) \cdot 2} \cdot \sigma_0(n,\chi_{-4}) \cdot \underbrace{\begin{cases} 2: & \left(\frac{-4}{n}\right) \neq -1; \\ 0: & \left(\frac{-4}{n}\right) = -1; \end{cases}}_{\text{local factor at 2}} = 8 \sum_{d|n} \left(\frac{-4}{d}\right),$$

and therefore

$$\#\Big\{(a,b) \in \mathbb{Z}^2 : \, a^2+b^2 = n\Big\} = 4 \sum_{d|n} \left(\frac{-4}{d}\right)$$
$$= 4 \cdot \Big(\#\{\text{divisors}\, d = 4\ell+1 \,\text{of}\, n\} - \#\{\text{divisors}\, d = 4\ell+3 \,\text{of}\, n\}\Big).$$

Remark 14. Experimentally one often finds that the weight-1 Eisenstein series attached to a discriminant form equals a theta series even in cases where it is impossible to associate a weight 1 theta series to the discriminant form in a meaningful sense; such relations are almost certainly coincidence resulting from small cusp spaces in weight 1. For example, the indefinite lattice with Gram matrix

$$S = \begin{pmatrix} 2 & -1 & -1 & -1 \\ -1 & 2 & -1 & -1 \\ -1 & -1 & 2 & -1 \\ -1 & -1 & -1 & 2 \end{pmatrix}$$

yields an Eisenstein series in which the component of $\mathfrak{e}_0$ is

$$E_1^*(\tau,0) = \frac{2}{3} + 4q + 4q^3 + 4q^4 + 8q^7 + 4q^9 + ...$$

i.e. $\frac{2}{3}$ times the theta series of the quadratic form x^2+xy+y^2. However, the discriminant form of S has signature 2 mod 8 and is therefore not represented by a negative-definite lattice whose theta series has weight one.

On the other hand, replacing S by

$$-3S = \begin{pmatrix} -6 & 3 & 3 & 3 \\ 3 & -6 & 3 & 3 \\ 3 & 3 & -6 & 3 \\ 3 & 3 & 3 & -6 \end{pmatrix}$$

yields an Eisenstein series in which the component of $\mathfrak{e}_0$ is

$$E_1^*(\tau,0) = \frac{34}{27} - \frac{4}{9}q + \frac{68}{9}q^3 - \frac{4}{9}q^4 - \frac{8}{9}q^7 + \frac{68}{9}q^9 \pm ...$$

with the surprising property that its coefficients have infinitely many sign changes; in particular, this example should make clear that $E_1^*(\tau,0)$ is not simply a theta series for every lattice.

5. Weight 3/2

In weight 3/2, the L-series term is

$$\tilde{L}(n,\gamma,3/2+e/2+2s) = \begin{cases} \frac{L(1+2s,\chi_{\mathcal{D}})}{\zeta(4s+2)} \prod_{\mathrm{bad}\,p} \frac{1-p^{(e-3)/2-2s}}{1-p^{-2-4s}} L_p(n,\gamma,1/2+e/2+2s) : & n \neq 0; \\ \frac{\zeta(4s+1)}{\zeta(4s+2)} \prod_{\mathrm{bad}\,p} \frac{(1-p^{(e-3)/2-2s})(1-p^{-1-4s})}{1-p^{-2-4s}} L_p(n,\gamma,1/2+e/2+2s) : & n = 0; \end{cases}$$

and it is holomorphic in $s=0$ unless $n=0$ or

$$\mathcal{D} = -2nd_\gamma^2|\Lambda'/\Lambda| \prod_{\mathrm{bad}\,p} p^2$$

is a square. In these cases, $\tilde{L}(n,\gamma,3/2+e/2+2s)$ has a simple pole with residue

$$\frac{3}{\pi^2} \prod_{\mathrm{bad}\,p} \lim_{s\to 0} \frac{(1-p^{e/2-3/2-2s})(1-p^{-1})}{1-p^{-2}} L_p(n,\gamma,1/2+e/2+2s)$$

if $n \neq 0$, and

$$\frac{3}{2\pi^2} \prod_{\mathrm{bad}\,p} \lim_{s\to 0} \frac{(1-p^{e/2-3/2-2s})(1-p^{-1})}{1-p^{-2}} L_p(n,\gamma,1/2+e/2+2s)$$

if $n=0$.

This pole cancels with the zero of $I(k,y,n,s)$ at $s=0$, whose derivative there is

$$\frac{d}{ds}\Big|_{s=0} I(k,y,n,s) = -16\pi^2(1+i)y^{-1/2}\beta(4\pi|n|y), \quad \text{where } \beta(x) = \frac{1}{16\pi}\int_1^\infty u^{-3/2}e^{-xu}\,\mathrm{d}u,$$

as calculated in [11], section 2.2. This expression is also valid for $n=0$, where it reduces to

$$\frac{d}{ds}\Big|_{s=0} I(k,y,0,s) = 2\pi(-i)^{3/2}\frac{d}{ds}\Big|_{s=0} 2^{-1/2-2s}y^{-1/2-s}\frac{\Gamma(2s+1/2)}{\Gamma(s)\Gamma(s+3/2)} = -\frac{2\pi}{\sqrt{y}}(1+i).$$

Therefore, $E^*_{3/2}(\tau,0)$ is a harmonic weak Maass form that is not generally holomorphic:

$$\begin{aligned} &E^*_{3/2}(\tau,0) \\ &= E_{3/2}(\tau) + \frac{3(-1)^{(3+b^+-b^-)/4}\sqrt{2}}{\pi\sqrt{y|\Lambda'/\Lambda|}}\Big(\sum_{\substack{\gamma\in\Lambda'/\Lambda \\ q(\gamma)\in\mathbb{Z}}} \prod_{p|\#(\Lambda'/\Lambda)} \frac{1-p^{(e-3)/2}}{1+p^{-1}} L_p(0,\gamma,1/2+e/2)\mathfrak{e}_\gamma\Big) \\ &\quad + \frac{48(-1)^{(3+b^+-b^-)/4}\sqrt{2}}{\sqrt{y|\Lambda'/\Lambda|}} \sum_{\substack{\gamma\in\Lambda'/\Lambda \\ n\in\mathbb{Z}-q(\gamma) \\ -2n|\Lambda'/\Lambda|=\square}} \Big[\beta(4\pi|n|y)\prod_{\mathrm{bad}\,p}\frac{1-p^{(e-3)/2}}{1+p^{-1}}L_p(n,\gamma,1/2+e/2)\Big]q^n\mathfrak{e}_\gamma, \end{aligned}$$

where $-2n|\Lambda'/\Lambda| = \square$ means that $-2n|\Lambda'/\Lambda|$ should be a rational square. (In particular, the real-analytic correction involves only exponents $n \leq 0$.)

Example 15. Zagier's Eisenstein series [11] occurs as the Eisenstein series for the quadratic form $q(x) = x^2$. The underlying harmonic weak Maass form is

$$\begin{aligned} E^*_{3/2}(\tau,0) &= E_{3/2}(\tau) - \frac{3}{\pi\sqrt{y}}\mathfrak{e}_0 - \frac{48}{\sqrt{y}} \sum_{\gamma\in\Lambda'/\Lambda} \sum_{\substack{n\in\mathbb{Z}-q(\gamma) \\ -n=\square}} \beta(4\pi|n|y) \underbrace{\prod_{\text{bad } p} \frac{1-p^{-1}}{1+p^{-1}} L_p(n,\gamma,1)}_{=1} q^n \mathfrak{e}_\gamma \\ &= E_{3/2}(\tau) - \frac{24}{\sqrt{y}} \sum_{n=-\infty}^{\infty} \beta(4\pi(n/2)^2 y) q^{-(n/2)^2} \mathfrak{e}_{n/2}. \end{aligned}$$

The coefficient of $q^{n/4}$ in

$$E_{3/2}(\tau) = \Big(1 - 6q - 12q^2 - 16q^3 - ...\Big)\mathfrak{e}_0 + \Big(-4q^{3/4} - 12q^{7/4} - 12q^{11/4} - ...\Big)\mathfrak{e}_{1/2}$$

is -12 times the Hurwitz class number $H(n)$. (We obtain Zagier's Eisenstein series in its usual form by summing the components, replacing τ by 4τ and y by $4y$, and dividing by -12.)

Remark 16. We can use essentially the same argument as Hirzebruch and Zagier [11] to derive the transformation law of the general $E_{3/2}$. Write $E^*_{3/2}(\tau,0)$ in the form

$$E^*_{3/2}(\tau,0) = E_{3/2} + \frac{1}{\sqrt{y}} \sum_{\gamma\in\Lambda'/\Lambda} \sum_{\substack{n\in\mathbb{Z}-q(\gamma) \\ n\le 0}} a(n,\gamma)\beta(-4\pi ny) q^n \mathfrak{e}_\gamma$$

with coefficients $a(n,\gamma)$. Applying the ξ-operator $\xi = y^{3/2}\overline{\frac{\partial}{\partial\overline{\tau}}}$ of [3] to $E^*_{3/2}(\tau,0)$ and using

$$\frac{d}{dy}\Big[\frac{1}{\sqrt{y}}\beta(y)\Big] = \frac{1}{16\pi}\frac{d}{dy}\Big[\int_y^\infty v^{-3/2}e^{-v}\,\mathrm{d}v\Big] = -\frac{1}{16\pi}y^{-3/2}e^{-y}$$

shows that the "shadow"

$$\vartheta(\tau) = \sum_{\gamma,n} a(n,\gamma)q^{-n}\mathfrak{e}_\gamma$$

is a modular form of weight 1/2 for the representation ρ (not its dual!), and

$$\begin{aligned} E^*_{3/2}(\tau,0) - E_{3/2}(\tau) &= y^{-1/2} \sum_{\gamma\in\Lambda'/\Lambda} \sum_{n\in\mathbb{Z}-q(\gamma)} a(n,\gamma)\beta(-4\pi ny)q^n\mathfrak{e}_\gamma \\ &= \frac{1}{16\pi}y^{-1/2}\int_1^\infty \sum_{\gamma,n} u^{-3/2}e^{-4\pi nuy}q^n\mathfrak{e}_\gamma\,\mathrm{d}u \\ &= \frac{1}{16\pi}y^{-1/2}\int_1^\infty u^{-3/2}\vartheta(2iuy-\tau)\,\mathrm{d}u \\ &= \frac{\sqrt{2i}}{16\pi}\int_{-x+iy}^{i\infty}(v+\tau)^{-3/2}\vartheta(v)\,\mathrm{d}v, \quad v = 2iuy-\tau. \end{aligned}$$

For any $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Mp_2(\mathbb{Z})$, defining $\tilde{M} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$ as in remark 6 and substituting $v = \tilde{M} \cdot t$ gives

$$\begin{aligned} E^*_{3/2}(M \cdot \tau, 0) - E_{3/2}(M \cdot \tau) &= \frac{\sqrt{2i}}{16\pi} \int_{-\overline{M\cdot\tau}}^{i\infty} \Big(\frac{a\tau + b}{c\tau + d} + v \Big)^{-3/2} \vartheta(v)\,\mathrm{d}v \\ &= \frac{\sqrt{2i}}{16\pi} \int_{-\overline{\tau}}^{d/c} \Big(\frac{a\tau + b}{c\tau + d} + \frac{at - b}{-ct + d} \Big)^{-3/2} \vartheta(\tilde{M} \cdot t) \frac{\mathrm{d}t}{(ct - d)^2} \\ &= \frac{\sqrt{2i}}{16\pi} (c\tau + d)^{3/2} \int_{-\overline{\tau}}^{d/c} (\tau + t)^{-3/2} \rho(\tilde{M}) \vartheta(t)\,\mathrm{d}t \\ &= (c\tau + d)^{3/2} \rho^*(M) \Big[\frac{\sqrt{2i}}{16\pi} \int_{-\overline{\tau}}^{d/c} (\tau + t)^{-3/2} \vartheta(t)\,\mathrm{d}t \Big]. \end{aligned}$$

Since $E^*_{3/2}(M \cdot \tau, 0) = (c\tau + d)^{3/2} \rho^*(M) E^*_{3/2}(\tau, 0)$, we conclude that

$$E_{3/2}(M \cdot \tau) = (c\tau + d)^{3/2} \rho^*(M) \Big[E_{3/2}(\tau) + \frac{\sqrt{2i}}{16\pi} \int_{d/c}^{i\infty} (\tau + t)^{-3/2} \vartheta(t)\,\mathrm{d}t \Big]. \tag{4}$$

Remark 17. The transformation law (4) can be used to give an easier sufficient condition for when $E_{3/2}$ is actually a modular form. For example, one can show that $M_{1/2}(\rho) = 0$ for the quadratic form $q(x, y, z) = -x^2 - y^2 - z^2$, which implies that the series ϑ defined above must be identically 0 and therefore

$$E_{3/2}(M \cdot \tau) = (c\tau + d)^{3/2} \rho^*(M) E_{3/2}(\tau),$$

so $E_{3/2}$ is a true modular form. (In this case, the local L-series $L_2(n, \gamma, 2 + 2s)$ at $p = 2$ is holomorphic at $s = 0$, and therefore the factor $(1 - 2^{-2s})$ annihilates the L-series term $\tilde{L}(n, \gamma, 3/2 + e/2)$ in the shadow.) This must be the theta series because $M_{3/2}(\rho^*)$ is one-dimensional.

It may be worth pointing out that the coefficient formulas ([5], theorem 4.8) for this theta series and for the Zagier Eisenstein series are nearly identical, since the squarefree parts of their discriminant and the "bad primes" are the same: the only real difference between them is the local factor at 2. For odd integers n, the local factor at 2 is easily computed and in both cases depends only on the remainder of n mod 8, so the coefficients $r_3(n)$ of the theta series and $H(4n)$ of the Zagier Eisenstein series within these congruence classes are proportional. Specifically,

$$r_3(n) = 12H(4n),\ n \equiv 1, 5\,(8);\ \ r_3(n) = 6H(4n),\ n \equiv 3\,(8);\ \ r_3(n) = 0,\ n \equiv 7\,(8).$$

These identities are well-known and were already proved by Gauss.

Example 18. Even when $M_{1/2}(\rho) \neq 0$, we can identify ϑ in $M_{1/2}(\rho)$ by computing finitely many coefficients. Let q be the quadratic form $q(x, y, z) = x^2 + y^2 - z^2$. The space $M_{1/2}(\rho)$ is always spanned by unary theta series embedded into $\mathbb{C}[\Lambda'/\Lambda]$

(as proven by Skoruppa [14]) and in this case one can find the basis

$$\begin{aligned}\vartheta_1(\tau) &= \Big(1+2q+2q^4+...\Big)(\mathfrak{e}_{(0,0,0)}+\mathfrak{e}_{(1/2,0,1/2)})\\ &\quad+\Big(2q^{1/4}+2q^{9/4}+2q^{25/4}+...\Big)(\mathfrak{e}_{(0,1/2,0)}+\mathfrak{e}_{(1/2,1/2,1/2)}),\\ \vartheta_2(\tau) &= \Big(1+2q+2q^4+...\Big)(\mathfrak{e}_{(0,0,0)}+\mathfrak{e}_{(0,1/2,1/2)})\\ &\quad+\Big(2q^{1/4}+2q^{9/4}+2q^{25/4}+...\Big)(\mathfrak{e}_{(1/2,0,0)}+\mathfrak{e}_{(1/2,1/2,1/2)}).\end{aligned}$$

The local L-series at the bad prime $p=2$ for the constant term $n=0$ are

$$(1-2^{-2s})L_p(0,0,2+2s) = \frac{1}{1-2^{-1-4s}} \quad \text{and} \quad (1-2^{-2s})L_p(0,\gamma,2+2s)=1$$

for $\gamma \in \{(1/2,0,1/2),(0,1/2,1/2)\}$, which implies that

$$E^*_{3/2}(\tau,0) = E_{3/2}(\tau) - \frac{3}{2\pi\sqrt{y}}\Big(\frac{4}{3}\mathfrak{e}_{(0,0,0)}+\frac{2}{3}\mathfrak{e}_{(1/2,0,1/2)}+\frac{2}{3}\mathfrak{e}_{(0,1/2,1/2)}\Big)+\Big(\text{negative powers of } q\Big)$$

and therefore the shadow in equation (6) must be

$$\vartheta(\tau) = -8\Big(\vartheta_1(\tau)+\vartheta_2(\tau)\Big).$$

In particular, the $\mathfrak{e}_0$-component $E_{3/2}(\tau)_0$ of $E_{3/2}(\tau)$ is a mock modular form of level 4 that transforms under $\Gamma(4)$ by

$$E_{3/2}(M\cdot\tau)_0 = (c\tau+d)^{3/2}\Big[E_{3/2}(\tau)_0 - \frac{\sqrt{2i}}{\pi}\int_{d/c}^{i\infty}(\tau+t)^{-3/2}\Theta(t)\,\mathrm{d}t\Big], \quad \Theta(t) = \sum_{n\in\mathbb{Z}}\mathbf{e}(n^2t).$$

It was shown by Bringmann and Lovejoy [1] that the series

$$\overline{\mathcal{M}}(\tau+1/2) = 1-\sum_{n=1}^{\infty}|\overline{\alpha}(n)|q^n = 1-2q-4q^2-8q^3-10q^4-...$$

of example 4, where $|\overline{\alpha}(n)|$ counts overpartition rank differences of n, has the same transformation behavior under the group $\Gamma_0(16)$, which implies that the difference between $\overline{\mathcal{M}}(\tau+1/2)$ and the $\mathfrak{e}_0$-component of $E_{3/2}$ is a true modular form of level 16. We can verify that these are the same by comparing all Fourier coefficients up to the Sturm bound.

6. Weight 2

In weight $k=2$, the L-series term is

$$\tilde{L}(n,\gamma,2+e/2+2s) = \begin{cases} \frac{1}{L(1+2s,\chi_D)}\prod_{\text{bad }p}(1-p^{e/2-2-2s})L_p(n,\gamma,1+e/2+2s): & n\neq 0;\\ \frac{L(1+2s,\chi_D)}{L(2+2s,\chi_D)}\prod_{\text{bad }p}(1-p^{e/2-2-2s})L_p(n,\gamma,1+e/2+2s): & n=0.\end{cases}$$

Since $L(1,\chi)$ is never zero for any Dirichlet character, the only way a pole can occur at $s=0$ is if $n=0$ and $D=|\Lambda'/\Lambda|$ is square. (In particular, when $|\Lambda'/\Lambda|$ is not square, E_2 is a modular form.)

Assume that $|\Lambda'/\Lambda|$ is square. Then
$$L(1+2s,\chi_D) = \zeta(1+2s)\prod_{\text{bad } p}(1-p^{-1-2s}),$$
and therefore $\tilde{L}(0,\gamma,2+e/2+2s)$, has a simple pole at $s=0$ with residue
$$\begin{aligned}&\mathrm{Res}\Big(\tilde{L}(0,\gamma,2+e/2+2s), s=0\Big)\\ &= \frac{1}{2L(2,\chi_D)}\prod_{\text{bad } p}\Big[(1-p^{-1})\lim_{s\to 0}(1-p^{e/2-2-2s})L_p(0,\gamma,1+e/2+2s)\Big]\\ &= \frac{3}{\pi^2}\lim_{s\to 0}\prod_{\text{bad } p}\frac{1-p^{e/2-2-2s}}{1+p^{-1}}L_p(0,\gamma,1+e/2+2s)\end{aligned}$$
for any $\gamma \in \Lambda'/\Lambda$ with $q(\gamma)\in\mathbb{Z}$. This pole is canceled by the zero of $I(2,y,0,s)$ at $s=0$ which has derivative
$$\begin{aligned}\frac{d}{ds}\Big|_{s=0} I(2,y,0,s) &= -2\pi(2y)^{-1}\frac{d}{ds}\Big|_{s=0}(2y)^{-2s}\frac{\Gamma(2s+1)}{\Gamma(s)\Gamma(s+2)}\\ &= -\frac{\pi}{y},\end{aligned}$$
so

(5)
$$E_2^*(\tau,0) = E_2(\tau) - \frac{3}{\pi y\sqrt{|\Lambda'/\Lambda|}}\lim_{s\to 0}\sum_{\substack{\gamma\in\Lambda'/\Lambda\\ q(\gamma)\in\mathbb{Z}}}\prod_{\text{bad } p}\frac{1-p^{e/2-2-2s}}{1+p^{-1}}L_p(0,\gamma,1+e/2+2s)\mathfrak{e}_\gamma.$$

Example 19. Let Λ be a unimodular lattice. The only bad prime is $p=2$. Using the hyperbolic plane $q(x,y)=xy$ to define Λ, the local L-function is
$$L_2(0,0,s) = \frac{1-2^{-s}}{(1-2^{1-s})^2}$$
with $L_2(0,0,2)=3$, so we obtain the well-known result
$$E_2^*(\tau,0) = E_2(\tau) - \frac{3}{\pi y}\cdot\frac{1-1/2}{1+1/2}L_2(0,0,2) = E_2(\tau) - \frac{3}{\pi y}.$$

Remark 20. We can summarize the above by saying that
$$E_2^*(\tau,0) = E_2(\tau) - \frac{1}{y}\sum_{\substack{\gamma\in\Lambda'/\Lambda\\ q(\gamma)\in\mathbb{Z}}}A(\gamma)\mathfrak{e}_\gamma$$
is a Maass form for some constants $A(\gamma)$. For $M = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in Mp_2(\mathbb{Z})$, since
$$E_2^*(M\cdot\tau,0) = (c\tau+d)^2\rho^*(M)E_2^*(\tau,0),$$
we find the transformation law
$$\begin{aligned}E_2(M\cdot\tau) &= E_2^*(M\cdot\tau,0) + \frac{|c\tau+d|^2}{y}\sum_{\substack{\gamma\in\Lambda'/\Lambda\\ q(\gamma)\in\mathbb{Z}}}A(\gamma)\mathfrak{e}_\gamma\\ &= (c\tau+d)^2\Big[\rho^*(M)E_2(\tau) - 2ic(c\tau+d)\sum_{q(\gamma)\in\mathbb{Z}}A(\gamma)\rho^*(M)\mathfrak{e}_\gamma\Big].\end{aligned}$$

Example 21. The weight-2 Eisenstein series for the quadratic form $q(x,y) = x^2 + 3xy + y^2$ is a true modular form because the discriminant 5 of q is not a square. In particular, the $\mathfrak{e}_0$-component

$$1 - 30q - 20q^2 - 40q^3 - 90q^4 - 130q^5 - 60q^6 - 120q^7 - 100q^8 - 210q^9 - ...$$

is a modular form of weight 2 for the congruence subgroup $\Gamma_1(5)$. Using remark 11, we see that the coefficient $c(n)$ of q^n for n coprime to 10 is

$$c(n) = \begin{cases} -30 \sum_{d|n} \left(\frac{5}{n/d}\right) d : & n \equiv \pm 1 \bmod 10; \\ -20 \sum_{d|n} \left(\frac{5}{n/d}\right) d : & n \equiv \pm 3 \bmod 10; \end{cases}$$

with a more complicated expression for other n involving the local factors at 2 and 5.

Example 22. The weight-2 Eisenstein series for the quadratic form $q(x,y) = 2xy$ is

$$\begin{aligned} E_2(\tau) = &\Big(1 - 8q - 40q^2 - 32q^3 - 104q^4 - ...\Big)\mathfrak{e}_{(0,0)} \\ &+ \Big(- 16q - 32q^2 - 64q^3 - 64q^4 - 96q^5 - ...\Big)(\mathfrak{e}_{(0,1/2)} + \mathfrak{e}_{(1/2,0)}) \\ &+ \Big(- 8q^{1/2} - 32q^{3/2} - 48q^{5/2} - 64q^{7/2} - 104q^{9/2} - ...\Big)\mathfrak{e}_{(1/2,1/2)} \\ = &\Big(1 - 8\sum_{n=1}^{\infty} \Big[\sum_{d|2n} (-1)^d d\Big] q^n\Big)\mathfrak{e}_{(0,0)} \\ &+ \Big(- 8\sum_{n=1}^{\infty} \Big[\sum_{d|n} (1 - (-1)^{n/d})d\Big] q^n\Big)(\mathfrak{e}_{(0,1/2)} + \mathfrak{e}_{(1/2,0)}) \\ &+ \Big(- 8\sum_{n=0}^{\infty} \sigma_1(2n+1)q^{n+1/2}\Big)\mathfrak{e}_{(1/2,1/2)}. \end{aligned}$$

It is not a modular form. On the other hand, the real-analytic correction (7) only involves the components $\mathfrak{e}_\gamma$ for which $q(\gamma) \in \mathbb{Z}$, i.e. $\mathfrak{e}_{(0,0)}, \mathfrak{e}_{(0,1/2)}, \mathfrak{e}_{(1/2,0)}$, so the components

$$1 - 8\sum_{n=1}^{\infty} \Big[\sum_{d|2n} (-1)^d d\Big] q^n, \;\; \sum_{n=1}^{\infty} \Big[\sum_{d|n} (1 - (-1)^{n/d})d\Big] q^n$$

are only quasimodular forms of level 4, while $\sum_{n=0}^{\infty} \sigma_1(2n+1)q^{2n+1}$ is a true modular form.

Example 23. Although the discriminant group of the quadratic form $q(x_1, x_2, x_3, x_4) = -x_1^2 - x_2^2 - x_3^2 - x_4^2$ has square order 16, the correction term still vanishes in this case. This is because the local L-functions for $p = 2$,

$$L_2(0, \gamma, 3 + s) = \begin{cases} \frac{2+2^{-s}}{2-2^{-s}} : & \gamma = 0; \\ 1 : & \gamma = (1/2, 1/2, 1/2, 1/2); \end{cases}$$

are both holomorphic at $s = 0$ and therefore annihilated by the term $(1-p^{4/2-2-2s})$ at $s = 0$. (Another way to see this is that $\sum_{\substack{\gamma \in \Lambda'/\Lambda \\ q(\gamma) \in \mathbb{Z}}} A(\gamma)\mathfrak{e}_\gamma$ is invariant under ρ due to the transformation law of E_2, but there are no nonzero invariants of ρ in this case.) In fact, the Eisenstein series E_2 for this lattice is exactly the theta series as one can see by calculating the first few coefficients. Comparing coefficients of the $\mathfrak{e}_0$-component leads immediately to Jacobi's formula:

$$\begin{aligned}
&\#\{(a,b,c,d) \in \mathbb{Z}^4 :\, a^2+b^2+c^2+d^2 = n\} \\
&= \frac{(2\pi)^2}{L(2,\chi_{64})\cdot 4} \cdot \sigma_1(n,\chi_{64}) \cdot L_2(n,0,3) \\
&= 8 \cdot \Big[\sum_{d|n} \Big(\frac{4}{n/d}\Big) d \Big] \cdot \begin{cases} 1 : & n \,\text{odd}; \\ 3 \cdot 2^{-v_2(n)} : & n \,\text{even}; \end{cases} \\
&= \begin{cases} 8 \sum_{d|n} d : & n\,\text{odd}; \\ 24 \sum_{\substack{d|n \\ d\,\text{odd}}} d : & n\,\text{even}; \end{cases}
\end{aligned}$$

for all $n \in \mathbb{N}$.

7. Weight 1/2

The Fourier expansion (1) is no longer valid in weight $k = 1/2$; in fact, the L-series factor in this case is

$$\tilde{L}\Big(n,\gamma,\frac{e+1}{2}+2s\Big) = \begin{cases} \frac{L_{\mathcal{D}}(2s)}{\zeta(4s)} \prod_{\mathrm{bad}\,p} \frac{1-p^{(e-1)/2-2s}}{1-p^{-4s}} L_p\Big(n,\gamma,\frac{e-1}{2}+2s\Big) : & n \neq 0; \\ \frac{\zeta(4s-1)}{\zeta(4s)} \prod_{\mathrm{bad}\,p} \frac{(1-p^{(e-1)/2-2s})(1-p^{1-4s})}{1-p^{-4s}} L_p\Big(n,\gamma,\frac{e-1}{2}+2s\Big) : & n = 0; \end{cases}$$

which generally has a singularity at $s = 0$, so our approach fails in weight 1/2.

Despite this, the weight 1/2 Eisenstein series $E^*_{1/2}(\tau,s)$ extends analytically to $s = 0$ (by a result of Shimura [13], it is entire except for a possible simple pole at $s = 1/2$). One way to study $E^*_{1/2}(\tau,s)$ is by applying the Bruinier-Funke operator $\xi_{3/2}$ to the weight 3/2 series $E^*_{3/2}(\tau,s)$ for the dual representation (i.e. the same lattice with negated quadratic form); from $\xi_{3/2} y^s = -\frac{si}{2} y^{s+1/2}$ one obtains $\xi_{3/2} E^*_{3/2}(\tau,s) = -\frac{si}{2} E^*_{1/2}(\tau,s+1/2)$ for all large enough s. Carrying over the arguments from the scalar-valued case (e.g. [8], section 4.10) should imply that $E^*_{1/2}(\tau,s)$ will satisfy some functional equation relating $E^*_{1/2}(\tau,s+1/2)$ to $E^*_{1/2}(\tau,-s)$ (or more likely some combination of $E^*_{1/2,\beta}(\tau,-s)$ as β runs through elements of Λ'/Λ with $q(\beta) \in \mathbb{Z}$ in general) although in the half-integer case this seems less straightforward. Assuming this, for large enough $\mathrm{Re}[s]$ it follows that $E^*_{1/2}(\tau,-s)$ should be a linear combination of $\xi_{3/2} E^*_{3/2,\beta}(\tau,s)$ with coefficients depending on s but independent of τ; we might even expect this to hold for arbitrary s and therefore conjecture:

Conjecture 24. *The zero-value $E^*_{1/2}(\tau,0)$ for a discriminant form $(\Lambda'/\Lambda, q)$ is a holomorphic modular form of weight 1/2; moreover it is a linear combination of the shadows of mock Eisenstein series $E_{3/2,\beta}(\tau)$ for $(\Lambda'/\Lambda, -q)$.*

Unfortunately, if this is true then from the point of view of this note there is little motivation to consider $E^*_{1/2}(\tau, 0)$ further: modular forms of weight 1/2 are spanned by what are essentially unary theta series and any resulting identities among coefficients will be uninteresting. There may be interest in higher terms of the Taylor expansion of $E^*_{1/2}(\tau, s)$ in the variable s which might be used to generate mock modular forms of weight 1/2 and higher depth, but this is outside the scope of this note.

There is one class of examples where this conjecture can be verified directly. In dimension $e = 1$, where the quadratic form is $q(x) = -mx^2$ for some $m \in \mathbb{N}$, we can make sense of the coefficient formula because the terms $1 - p^{1-4s}$ are cancelled by the numerators at $s = 0$, and the Fourier series then provides the analytic continuation of $E^*_{1/2}(\tau, s)$ to $s = 0$. The L-series factor in this case is

$$\tilde{L}(n, \gamma, 1 + 2s) = \begin{cases} \frac{L(2s,\chi_{\mathcal{D}})}{\zeta(4s)} \prod_{\mathrm{bad}\, p} \frac{L_p(n,\gamma,2s)}{1+p^{-2s}} : & n \neq 0; \\ \frac{\zeta(4s-1)}{\zeta(4s)} \prod_{\mathrm{bad}\, p} \frac{(1-p^{1-4s})L_p(n,\gamma,2s)}{1+p^{-2s}} : & n = 0. \end{cases}$$

Here, $\mathcal{D}$ is the discriminant

$$\mathcal{D} = 2d_\gamma^2 n |\Lambda'/\Lambda| \prod_{\mathrm{bad}\, p} p^2 = 4mnd_\gamma^2 \prod_{\mathrm{bad}\, p} p^2.$$

Suppose for simplicity that m is squarefree (and in particular, $\beta = 0$ is the only element of Λ'/Λ with $q(\beta) \in \mathbb{Z}$). The local L-factors can be calculated by elementary means (for example, with Hensel's lemma), and the result in this case is that $E_{1/2}(\tau) = 1 \cdot \mathfrak{e}_0 + \sum_{\gamma \in \Lambda'/\Lambda} \sum_{n \in \mathbb{Z} - q(\gamma)} c(n, \gamma) q^n \mathfrak{e}_\gamma$ with

$$c(n, \gamma) = 2 \cdot (1/2)^\varepsilon, \;\; \varepsilon = \#\{\text{primes } p \neq 2 \text{ dividing } d_\gamma\} + \begin{cases} 1 : & 4 | d_\gamma \\ 0 : & \text{otherwise.} \end{cases}$$

Here, d_γ is the denominator of γ; that is, the smallest number for which $d_\gamma \gamma \in \Lambda$.

The shadow of the mock Eisenstein series $E_{3/2}(\tau)$ attached to mx^2 can be computed directly as well, although this is more difficult. On the other hand, one can use the following trick: via the theta decomposition, the nonholomorphic weight 3/2 Eisenstein series $E^*_{3/2}(\tau, 0)$ corresponds to a nonholomorphic, scalar Jacobi Eisenstein series $E^*_{2,m}(\tau, z, 0)$ of index m. The argument of chapter 4 of [9] still applies to this situation and in particular $E^*_{2,m}(\tau, z, 0) = \frac{1}{\sigma_1(m)} E^*_{2,1}(\tau, z, 0) | V_m$ for the Hecke-type operator

$$\Phi | V_m(\tau, z) = m \sum_M (c\tau + d)^{-2} \mathbf{e}\Big(- \frac{cmz^2}{c\tau + d}\Big) \Phi\Big(\frac{a\tau + b}{c\tau + d}, \frac{mz}{c\tau + d}\Big),$$

the sum taken over cosets of determinant-m integral matrices M by $SL_2(\mathbb{Z})$. (Here we must assume that m is squarefree). However, $E^*_{2,1}(\tau, z, 0)$ arises through the theta decomposition from the Zagier Eisenstein series and so its coefficients are well-known. In this way one can compute

$$E^*_{3/2}(\tau, 0) = -\frac{12}{\sigma_1(m)} \sum_{\gamma \in \Lambda'/\Lambda} \sum_{n \in \mathbb{Z} - q(\gamma)} \sum_{a | m} aH(4mn/a^2) q^n \mathfrak{e}_\gamma + \frac{1}{\sqrt{y}} \sum_{\gamma \in \Lambda'/\Lambda} \sum_{n \in \mathbb{Z} + q(\gamma)} a(n, \gamma) q^{-n} \mathfrak{e}_\gamma,$$

where $H(n)$ is the Hurwitz class number (and $H(n) = 0$ if n is noninteger) and the coefficients of the shadow are

$$a(n, \gamma) = \begin{cases} -24\sqrt{m}\frac{\sigma_0(m)}{\sigma_1(m)} : & n = 0; \\ -48\sqrt{m}\frac{\sigma_0(\mathrm{gcd}(m,n))}{\sigma_1(m)} : & mn = \square,\ mn \neq 0; \\ 0 : & \text{otherwise}, \end{cases}$$

and where we use the convention $\mathrm{gcd}(m, n) = \prod_{v_p(m), v_p(n) \geq 0} p^{\min(v_p(m), v_p(n))}$ (e.g. $\mathrm{gcd}(30, 3/4) = 3$). Unraveling this, we see that $E_{1/2}(\tau)$ differs from the shadow of $E_{3/2}(\tau)$ by the factor $-24\sqrt{m}\frac{\sigma_0(m)}{\sigma_1(m)}$.

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