CONES OF NOETHER–LEFSCHETZ DIVISORS AND MODULI SPACES OF HYPERKÄHLER MANIFOLDS

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Abstract. We give a general formula for generators of the NL-cone, the cone of effective linear combinations of irreducible components of Noether-Lefschetz divisors, on an orthogonal modular variety. We then fully describe the NL-cone and its extremal rays in the cases of moduli spaces of polarized K3 surfaces and hyperkähler manifolds of known deformation type for low degree polarizations. Moreover, we exhibit explicit divisors in the boundary of NL-cones for polarizations of arbitrarily large degrees. Additionally, we study the NL-positivity of the canonical class for these modular varieties. As a consequence, we obtain uniruledness results for moduli spaces of primitively polarized hyperkähler manifolds of OG6 and Kum_n-type. Finally, we show that any family of polarized hyperkähler fourfolds of Kum_2-type with polarization of degree 2 and divisibility 2 over a projective base is isotrivial.

Contents

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1. Introduction

In the classical study of moduli spaces of curves, modular divisors, namely divisors parametrizing curves satisfying special properties play an important role. For instance, the calculation of the class of Brill–Noether divisors $\overline{\mathcal{M}}^{d}_{g,r}$ parametrizing curves C of genus g that admit a g^r_d when $g-(r+1)(g-d+r)=-1$ was an essential step in Harris, Mumford, Eisenbud's proof that $\overline{M}_g$ is of general type for $g\geq 24$ [HM82, EH87].

In the case of the moduli space $\mathcal{F}_{2d}$ of quasi-polarized K3 surfaces of degree $2d$, the most natural source of modular divisors is Noether–Lefschetz divisors. A very general point $(S,H)\in\mathcal{F}_{2d}$ has Picard group $\mathrm{Pic}(S)=\mathbb{Z}H$ and so the locus in $\mathcal{F}_{2d}$ where $\rho(S)\geq 2$ is a countable union of divisors, called Noether–Lefschetz divisors (or NL divisors). Concretely, a Noether–Lefschetz divisor $\mathcal{D}_{h,a}$ on $\mathcal{F}_{2d}$ is the reduced divisor obtained by taking the closure of the locus of points $(S,H)\in\mathcal{F}_{2d}$ for which there exists a class $\beta\in\mathrm{Pic}(S)$, not proportional to H, with $\beta^2=2h-2$ and $\beta.H=a$.

Heegner divisors generalize Noether–Lefschetz divisors to arbitrary orthogonal modular varieties $\mathcal{D}/\Gamma$ by viewing Noether–Lefshetz divisors as images of hyperplane arrangements in $\mathcal{D}$ under the modular projection $\pi:\mathcal{D}\longrightarrow\mathcal{D}/\Gamma$. More precisely, let Λ be an even lattice of signature $(2,n)$ with bilinear form $\langle\cdot,\cdot\rangle$ (which extends to $\Lambda_{\mathbb{C}}$) and $\mathcal{D}_\Lambda$ the Type IV domain given by one of the two components (exchanged by complex conjugation) of

$$\left\{[Z]\in\mathbb{P}\left(\Lambda_{\mathbb{C}}\right)\middle|\langle Z,Z\rangle=0,\langle Z,\overline{Z}\rangle>0\right\}.$$

We assume that Γ is a finite index subgroup of $\widetilde{\mathrm{O}}^+(\Lambda)$, the group of orientation preserving isomorphisms of Λ acting trivially on the discriminant group $D(\Lambda_{2d})=\Lambda_{2d}^\vee/\Lambda$ and consider the quotient $\mathcal{D}_\Lambda/\Gamma$. The fundamental example to have in mind is when

$$\Lambda=\Lambda_{2d}=U^{\oplus 2}\oplus E_8(-1)^{\oplus 2}\oplus\mathbb{Z}\ell,\quad\text{with}\quad\langle\ell,\ell\rangle=-2d,$$

and $\Gamma=\widetilde{\mathrm{O}}^+(\Lambda_{2d})$. Then $\mathcal{F}_{2d}=\mathcal{D}_{\Lambda_{2d}}/\widetilde{\mathrm{O}}^+(\Lambda_{2d})$ is a coarse moduli space for primitively quasi-polarized K3 surfaces of degree $2d$.

For $v\in\Lambda_{\mathbb{Q}}$, one considers the hyperplane section $D_v=v^\perp\cap\mathcal{D}_\Lambda$. With the quadratic form $Q(v)=\frac{\langle v,v\rangle}{2}$, for a fixed class $\mu+\Lambda\in D(\Lambda)$ and $m\in Q(\mu)+\mathbb{Z}$ non-positive, the cycle $\sum_{\substack{v\in\mu+\Lambda\\ Q(v)=m}}D_v$ is Γ-invariant and descends to a $\mathbb{Q}$-Cartier divisor $H_{m,\mu}$ on $\mathcal{D}_\Lambda/\Gamma$, called a *Heegner divisor*. In general, $H_{m,\mu}$ is neither reduced, nor irreducible and its irreducible components, called *primitive Heegner divisors*, are denoted $P_{\Delta,\delta}$. In the K3 case, Heegner divisors are related to Noether–Lefschetz divisors via $\mathcal{D}_{h,a}=H_{-m,\mu}$ if $d\nmid a$ and $\mathcal{D}_{h,a}=\frac{1}{2}H_{-m,\mu}$ if $d\mid a$, where $m=\frac{a^2}{4d}-(h-1)$ and $\mu=a\ell_*$ for $\ell_*=\frac{\ell}{2d}\in D(\Lambda_{2d})$ the standard generator [MP13, Lemma 3].

Maulik–Pandharipande conjectured [MP13, Conjecture 3] that the rational Picard group $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{F}_{2d})$ is generated by Noether–Lefschetz divisors $\mathcal{D}_{h,a}$. Bergeron–Li–Millson–Moeglin proved a generalization of Maulik–Panharipande's conjecture, showing that, in fact, the Picard group with rational coefficients $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{D}_\Lambda/\Gamma)$ of any such orthogonal modular variety with $n \geq 3$ is generated by Heegner divisors [BLMM17]. The rank of $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{D}_\Lambda/\Gamma)$ was computed by Bruinier in [Bru02b].

An important invariant of an algebraic variety X is its cone of pseudo-effective divisors $\overline{\mathrm{Eff}}(X)$, which governs much of the birational geometry of X. The cone $\overline{\mathrm{Eff}}(X)$ is defined as the closure of the cone of effective $\mathbb{R}$-divisors on X. In general, it can be quite difficult to determine when $\overline{\mathrm{Eff}}(X)$ is finitely-generated, let alone compute it explicitly.

In the case of an orthogonal modular variety $\mathcal{D}_\Lambda/\Gamma$, a natural subcone of $\overline{\mathrm{Eff}}(\mathcal{D}_\Lambda/\Gamma)$ is the NL-cone $\mathrm{Eff}^{NL}(\mathcal{D}_\Lambda/\Gamma)$ of effective $\mathbb{R}$-linear combinations of primitive Heegner divisors on $\mathcal{D}_\Lambda/\Gamma$. The NL-cone contains the subcone $\mathrm{Eff}^{H}(\mathcal{D}_\Lambda/\Gamma)$ generated by the (non-primitive) Heegner divisors on $\mathcal{D}_\Lambda/\Gamma$. The study of NL-cones was initiated in [Pet15], in the case $\mathcal{D}_\Lambda/\Gamma = \mathcal{F}_{2d}$, where Peterson raised the following questions [Pet15, Section 4.5]:

(1) Is $\mathrm{Eff}^{NL}(\mathcal{F}_{2d})$ finitely-generated (polyhedral)?
(2) Can we compute generators for $\mathrm{Eff}^{NL}(\mathcal{F}_{2d})$?
(3) Is there an effective divisor not in $\mathrm{Eff}^{NL}(\mathcal{F}_{2d})$?

Bruinier–Möller [BM19] answered the first question affirmatively, showing that for a general orthogonal modular variety $X = \mathcal{D}_\Lambda/\widetilde{\mathrm{O}}^+(\Lambda)$ such that Λ is of signature $(2,n)$ and splits off two copies of the hyperbolic plane, the cone $\mathrm{Eff}^{NL}(X)$ is always polyhedral.

In this paper, we tackle Question (2) for $X = \mathcal{D}_\Lambda/\widetilde{\mathrm{O}}^+(\Lambda)$ under the same assumptions. We consider the $\mathbb{Q}$-vector space $S_{k,\Lambda}$ of vector-valued cusp forms of weight $k = 1 + \frac{n}{2}$ with respect to the Weil representation and the coefficient extraction functionals in $S_{k,\Lambda}^\vee$

$$c_{m,\mu} : S_{k,\Lambda} \longrightarrow \mathbb{Q}, \quad \sum a_{m,\mu} q^m \mathfrak{e}_\mu \mapsto a_{m,\mu}.$$

Let $b \geq \lceil k/12 \rceil$ be an integer such that the set of $c_{m,\mu}$ with $0 < m \leq b$ and $\mu \in D(\Lambda)$ generates $S_{k,\Lambda}^\vee$. Then, we consider the weakly holomorphic modular form

$$\Delta^{-b} \cdot E_{(2-k)+12b,\Lambda(-1)} = \sum_{\substack{(m,\mu) \\ -b \leq m}} \alpha_{m,\mu} q^m \mathfrak{e}_\mu, \tag{1}$$

where $\Delta(\tau)$ is the scalar-valued discriminant modular form and $E_{(2-k)+12b,\Lambda(-1)}$ is the Eisenstein series of weight $(2-k)+12b$ associated to $\Lambda(-1)$ (see Equation (5)).

Our first result is the following, with the explicit bounds in Theorems 3.4 and 3.6.

Theorem 1.1. *Let Λ be an even lattice of signature $(2,n)$ with $n \geq 3$ and $X = \mathcal{D}_\Lambda/\widetilde{\mathrm{O}}^+(\Lambda)$ its modular variety. Fixing b as above, there are explicit bounds Ξ and Ω, depending on k, the discriminant of Λ, and the $\alpha_{m,\mu}$ with $-b \leq m \leq 0$ in (1) , such that*

(1) The cone $\mathrm{Eff}^{H}(X)$ is generated by all $H_{-m,\mu}$ with $0 \leq m \leq \Xi$, when Λ splits off one copy of the hyperbolic plane

(2) The cone $\mathrm{Eff}^{NL}(X)$ is generated by all $P_{-\Delta,\delta}$ with $0 \leq \Delta \leq \Omega$, when Λ splits off two copies of the hyperbolic plane.

Theorem 1.1 together with its implementation in Sage package [Wila] enables the computation of $\mathrm{Eff}^{NL}(X)$ given any such Λ (see Section 1.1 below).

The proof of Theorem 1.1 relies on the relationship between Heegner divisors on X and vector-valued modular forms with respect to the Weil representation for Λ. In [BM19] the polyhedrality of the NL-cone is established by showing that the Hodge class λ lies on the interior of the NL-cone, and the rays $P_{\Delta,\delta}\mathbb{Q}_{\geq 0}$ converge to $\lambda\mathbb{Q}_{\geq 0}$ as Δ grows. Establishing a concrete list of generators of $\mathrm{Eff}^{NL}(X)$ amounts to making the convergence rate explicit which translates into bounding explicitly the growth of the coefficients of the relevant vector-valued modular forms (see Section 3). For vector-valued cusp forms of half-integer weight, despite the considerable literature on bounds for the growth of Fourier coefficients, we are unaware of a general bound with explicit constants. Using Poincaré series and Kloosterman sums we derive weak, yet explicit, bounds that suffice for our purposes.

1.1. **Explicit Computations of** $\mathrm{Eff}^{NL}(X)$**.** We then focus on cases where the quotient $X = \mathcal{D}_\Lambda/\widetilde{\mathrm{O}}^+(\Lambda)$ arises as (a finite cover of) a partial compactification of a coarse moduli space of polarized K3 surfaces or hyperkähler manifolds. We give explicit formulas for $\mathrm{Eff}^{NL}(X)$ in terms of generating rays for low-degree polarizations: see Table 1 for the case of (quasi)-polarized K3 surfaces, Tables 4 and 5 for the case of hyperkähler fourfolds of $\mathrm{K3}^{[2]}$-type, and Theorem 7.8 for the case of Kum_2-type hyperkähler manifolds. In the case of $\mathcal{F}_{2d}$, the calculations in Table 1 confirm (aside from one additional generator in the case $d = 13$) the predictions in [Pet15] who computed, for $d \leq 18$, the cone generated by the set of $8d$ generators $P_{\Delta,\delta}$, for $\delta \in D(\Lambda), \Delta \in Q(\delta) + s$ with $s = 0, 1, 2, 3$, and conjectured that this cone coincides with $\mathrm{Eff}^{NL}(\mathcal{F}_{2d})$.

In the case of hyperkähler fourfolds of $\mathrm{K3}^{[2]}$-type we also record the NL-positivity of the canonical class, noting that K_X lies inside $\mathrm{Eff}^{NL}(X)$ as soon as the polarization degree exceeds the lowest possible.

Note that a negative answer to Question (3) above would imply, via [BM19], the polyhedrality of $\overline{\mathrm{Eff}}(X)$. In contrast, a positive answer, would arise from exhibiting an effective divisor which is not an effective linear combination of primitive Heegner divisors. One possible approach in this direction would be to exhibit a big divisor lying on the boundary of $\mathrm{Eff}^{NL}(X)$. For this it is helpful to describe the boundary of $\mathrm{Eff}^{NL}(X)$.

1.2. **Boundary rays for arbitrarily high discriminant.** While the Sage package [Wila] enables the computation of generating rays of $\mathrm{Eff}^{NL}(X)$ for fixed Λ, the approach becomes computationally infeasible as the discriminant of the lattice Λ grows large. In Section 4, we study the behavior of the NL cones $\mathrm{Eff}^{NL}(\mathcal{D}_\Lambda/\Gamma)$ in relation to finite maps $\phi\colon \mathcal{D}/\Gamma' \to \mathcal{D}/\Gamma$ induced by a finite index embeddings $\Lambda' \subset \Lambda$. We show in Proposition 4.2 that the pullback along ϕ preserves both the boundary and the interior of $\mathrm{Eff}^{NL}(\mathcal{D}_\Lambda/\Gamma)$. In particular if Z is an element of $\partial\mathrm{Eff}^{NL}(\mathcal{D}/\Gamma)$, then every irreducible component of ϕ^*Z is contained in $\partial\mathrm{Eff}^{NL}(\mathcal{D}/\Gamma')$. Moreover, we provide an explicit formula in Proposition 4.5 for the pullback of a (non-primitive) Heegner divisor under such a map. In combination with the explicit computations discussed in 1.1, this enables one to exhibit explicit boundary rays of $\mathrm{Eff}^{NL}(X)$ as discriminant of the lattice Λ grows arbitrarily large.

We carry this out in more detail in the case $X = \mathcal{F}_{2d}$. There, for any $r > 0$, the embedding $\Lambda_{2dr^2} \hookrightarrow \Lambda_{2d}$ given by $\ell' \mapsto r\ell$ induces a finite map $\phi\colon \mathcal{F}_{2dr^2} \to \mathcal{F}_{2d}$. In Proposition 5.3, we give an explicit formula for the support of the pullback $\phi^* P_{m,\mu}$ of a primitive Heegner $P_{m,\mu} \in \mathrm{Eff}^{NL}(\mathcal{F}_{2d})$.

Example 1.2. In Table 2 we compute the pullback under $\phi\colon \mathcal{F}_{2r^2} \to \mathcal{F}_2$ of the two generating rays $P_{-1,0}$ and $P_{-\frac{1}{4},\ell_*}$ of $\mathrm{Eff}^{NL}(\mathcal{F}_2)$ when $2 \leq r \leq 8$. For instance when $r = 8$, we obtain that the components of $\phi^* P_{-1,0}$ are

$$P_{-1,0}, P_{-1,16\ell'_*}, P_{-1,32\ell'_*}, P_{-1,48\ell'_*}, P_{-1,64\ell'_*}, P_{-\frac{1}{9},4\ell'_*}.$$

This singles out some faces of the boundary of $\mathrm{Eff}^{NL}(\mathcal{F}_{128})$. On the other hand, the unigonal divisor $P_{-\frac{1}{4},\ell_*}$ in $\mathcal{F}_2$ is known [Sha80] to be extremal (in the boundary of the effective cone) and thus all components of its pullback under ϕ lie in the boundary of $\overline{\mathrm{Eff}}(\mathcal{F}_{2r^2})$. Thus our method produces explicit extremal faces for large d.

It is natural to ask whether in the K3 setting $P_{-1,0}$ is always in the boundary of the NL cone and if there is an example where it is extremal. Note that for moduli spaces of higher dimensional hyperkähler varieties $P_{-1,0}$ can be in the interior of the NL cone, see Theorem 7.8. The formula given in Proposition 5.3 together with the computations in Table 1 allow us for instance to conclude the following.

Theorem 1.3. *The primitive Heegner divisor $P_{-1,0}$ lies in the boundary of $\mathrm{Eff}^{NL}(\mathcal{F}_{2dr^2})$ for all $1 \leq d \leq 20$ and all $r > 0$.*

1.3. **Uniruledness results.** Mukai in a celebrated series of papers [Muk88, Muk92, Muk06, Muk10, Muk16] constructed unirational parameterizations of $\mathcal{F}_{2d}$ for low-degrees. This has been recently improved by Farkas–Verra in [FV18, FV21]. Much less is known for higher-dimensional hyperkähler varieties, where known unirational parameterizations are available only for a few cases, all of them for moduli spaces of hyperkähler varieties of $\mathrm{K3}^{[n]}$-type, cf. [BD85, O'G06, IR01, IR07, DV10, BLM$^+$21]. Constructing unirational parameterizations in low degree for moduli spaces of hyperkähler varieties of generalized Kummer and OG6-types is a challenge where, as far as we know, no single explicit construction is known. Here we consider the simpler problem of establishing uniruledness.

In Section 7, we consider the moduli spaces $\mathcal{M}^{\gamma}_{\mathrm{OG6},2d}$ and $\mathcal{M}^{\gamma}_{\mathrm{Kum}_n,2d}$, which are the period domain partial compactifications of the moduli spaces $\left(\mathcal{M}^{\gamma}_{\mathrm{OG6},2d}\right)^{\circ}$ and $\left(\mathcal{M}^{\gamma}_{\mathrm{Kum}_n,2d}\right)^{\circ}$ parameterizing primitively polarized hyperkähler sixfolds of OG6-type respectively $2n$-folds of Kum_n-type with a primitive polarization of degree $2d$ and divisibility γ. We remark that the moduli space $\mathcal{M}^{\gamma}_{\mathrm{OG6},2d}$ is always irreducible and in the case $\gamma = 2$ it is non-empty only when $d \equiv -1, -2 \mod 4$. Similarly, setting $d = 1$ and

$\gamma \in \{1,2\}$, the moduli space $\mathcal{M}^{\gamma}_{\mathrm{Kum}_n,2}$ is irreducible and in the case $\gamma = 2$ its nonempty only when $n \equiv 2 \mod 4$.

Theorems 7.2 and 7.5 establish the following uniruledness results:

Theorem 1.4. *The moduli space $\mathcal{M}^{\gamma}_{\mathrm{OG6},2d}$ is uniruled in the following cases*

(i) *when $\gamma = 1$ for $d \leq 12$,*

(ii) *when $\gamma = 2$ for $t \leq 10$ and $t = 12$ with $d = 4t - 1$,*

(iii) *when $\gamma = 2$ for $t \leq 9$ and $t = 11, 13$ with $d = 4t - 2$.*

The moduli spaces $\mathcal{M}^{1}_{\mathrm{Kum}_n,2}$ and $\mathcal{M}^{2}_{\mathrm{Kum}_n,2}$ are uniruled in the following cases:

(i) *when $\gamma = 1$ for $n \leq 15$ and $n = 17, 20$,*

(ii) *when $\gamma = 2$ for $t \leq 11$ and $t = 13, 15, 17, 19$, where $n = 4t - 2$.*

An immediate consequence of the work of H. Wang and the fourth author [WW21, Theorem 5.4] together with Lemmas 7.1 and 7.4, appearing here, is the rationality of $\mathcal{M}^{2}_{\mathrm{Kum}_2,2}$ and unirationality of $\mathcal{M}^{2}_{\mathrm{OG6},6}$ and $\mathcal{M}^{1}_{\mathrm{OG6},2}$.

Our approach to uniruledness is inspired by [Pet15]. The idea is to express the canonical class of (a smoothing of a toroidal compactification of) such a quotient $\mathcal{M} = \mathcal{D}_{\Lambda} \big/ \widetilde{\mathrm{O}}^{+}(\Lambda)$ in terms of Heegner divisors $H_{m,\mu}$. One then uses formulas of Kudla [Kud03] and Bruinier–Kuss [BK01] expressing the intersection of these Heegner divisors with the power $\lambda^{\dim \mathcal{M} - 1}$ of the Hodge class λ in terms of coefficients of an Eisenstein series in order to show that the intersection of the canonical class with $\lambda^{\dim \mathcal{M} - 1}$ is negative. Since $\lambda^{\dim \mathcal{M} - 1}$ is a covering curve (in particular nef) the canonical class is then not pseudo-effective and uniruledness follows from [BDPP13].

Lastly as part of Theorem 7.8 we establish

Theorem 1.5. *The moduli space $\left(\mathcal{M}^{2}_{\mathrm{Kum}_2,2}\right)^{\circ}$ parameterizing polarized hyperkähler fourfolds with polarization of degree 2 and divisibility 2 is quasi-affine.*

Motivated by the questions treated in [BKPSB98, DM22] (see [BKPSB98, Theorem 1.3]) an immediate consequence is:

Corollary 1.6. *Any family $f : \mathcal{X} \longrightarrow B$ over a projective base B of polarized hyperkähler fourfold deformation equivalent to Kum_2 with polarization of degree 2 and divisibility 2 is isotrivial.*

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2. Preliminaries

Let Λ be an even lattice of signature $(2,n)$ with bilinear form given by $\langle\cdot,\cdot\rangle$. The bilinear form extends to $\Lambda_{\mathbb{C}}$ and we call $\mathcal{D}_\Lambda$ one of the two components of

$$\left\{[Z]\in\mathbb{P}\left(\Lambda_{\mathbb{C}}\right)\,\middle|\,\langle Z,Z\rangle=0,\langle Z,\overline{Z}\rangle>0\right\}.$$

Further, we denote by Γ a finite index subgroup of the group $\mathrm{O}^+(\Lambda)$ of automorphisms of Λ fixing the component $\mathcal{D}_\Lambda$. The quotient of $\mathcal{D}_\Lambda$ by Γ is often referred to as an *orthogonal modular variety*. It is a quasi-projective variety [BB66] that for various choices of lattice Λ and arithmetic groups Γ turns out to be a partial compactification of a coarse moduli space of polarized varieties. The first case of interest in this paper is when

$$\Lambda_{2d}=U^{\oplus 2}\oplus E_8(-1)^{\oplus 2}\oplus\mathbb{Z}\ell,\quad\text{with}\quad\langle\ell,\ell\rangle=-2d$$

and when the arithmetic group $\Gamma=\widetilde{\mathrm{O}}^+(\Lambda)$ is the group of orientation preserving isomorphisms of Λ acting trivially on the discriminant group $D(\Lambda)=\Lambda^\vee/\Lambda$. The quotient

$$\mathcal{F}_{2d}=\mathcal{D}_{\Lambda_{2d}}/\widetilde{\mathrm{O}}^+(\Lambda_{2d})$$

is a coarse moduli space for quasi-polarized K3 surfaces (S,H), i.e., where H is primitive, big, and nef, of degree $H^2=2d$.

As mentioned in the introduction, a very general point $(S,H)\in\mathcal{F}_{2d}$ has Picard group $\mathrm{Pic}(S)=\mathbb{Z}H$, and a large source of geometric divisors comes from imposing the condition that the Picard rank jumps. These are *Noether–Lefschetz divisors*. There are different characterizations of these divisors: by keeping track of a rank two lattice embedding $L\hookrightarrow\mathrm{Pic}(S)$, by imposing the existence of an extra class $\beta\in\mathrm{Pic}(S)$ with fixed intersections $(\beta^2,\beta\cdot H)=(2h-2,a)$, and by looking at images of hyperplanes in $\mathcal{D}_{\Lambda_{2d}}$ via the quotient map

$$\pi_{2d}:\mathcal{D}_{\Lambda_{2d}}\longrightarrow\mathcal{F}_{2d}.$$

These are all equivalent approaches (see [MP13, Section 1 and Lemma 3]). In what follows, we focus on the third approach.

2.1. **Heegner and NL divisors.** We assume $\Gamma\subset\widetilde{\mathrm{O}}^+(\Lambda)$. Let $Q(x)=\frac{\langle x,x\rangle}{2}$ be the corresponding quadratic form. For fixed $v\in\Lambda^\vee\subset\Lambda_{\mathbb{Q}}$, we set

$$D_v=v^\perp\cap\mathcal{D}_\Lambda=\left\{[Z]\in\mathcal{D}_\Lambda\,|\,\langle Z,v\rangle=0\right\}.$$

Let $\mu+\Lambda\in\Lambda^\vee/\Lambda$ and $m\in Q(\mu)+\mathbb{Z}$ non-positive. Then the cycle

$$\sum_{\substack{v\in\mu+\Lambda\\ Q(v)=m}}D_v \tag{2}$$

is Γ-invariant and descends to a $\mathbb{Q}$-Cartier divisor $H_{m,\mu}$ called a *Heegner divisor*. In general, $H_{m,\mu}$ is neither reduced, nor irreducible. The existence of two vectors $v,v'\in\Lambda^\vee$

with the same square and discriminant class for which $D_v = D_{v'}$ is a source for non-reduced components of $H_{m,\mu}$. Similarly, several Γ-orbits of elements in $\Lambda^\vee$ with the same square and discriminant class give rise to several (possibly non-reduced) components.

Under the given assumption that $\Gamma \subset \widetilde{O}^+(\Lambda)$, all the components of $H_{m,\mu}$ have multiplicity two if $\mu = -\mu$ in $\Lambda^\vee/\Lambda$ and all have multiplicity one otherwise. Further, the line bundle $\mathcal{O}(-1)$ on $\mathcal{D}_\Lambda \subset \mathbb{P}(\Lambda_\mathbb{C})$ admits a natural Γ-action and descends to a $\mathbb{Q}$-line bundle λ called the *Hodge bundle*. One declares $H_{0,0} = -\lambda$.

In the K3 case $\mathcal{F}_{2d} = \mathcal{D}_{\Lambda_{2d}}/\widetilde{\mathrm{O}}^+(\Lambda_{2d})$, Noether-Lefschetz divisors are often described as the reduced divisor obtained by taking the closure of the locus

$$\mathcal{D}_{h,a} \subset \mathcal{F}_{2d}$$

of points (S,H) for which there exists a class $\beta \in \mathrm{Pic}(S)$ with $\beta^2 = 2h-2$ and $\beta \cdot H = a$. In this case [MP13, Lemma 3], if d does not divides a:

$$\mathcal{D}_{h,a} = H_{-m,\mu} \quad \text{with } m = \frac{a^2}{4d} - (h-1), \quad \text{and } \mu = a\ell_*.$$

Here $\ell_* = \frac{\ell}{2d} \in D(\Lambda_{2d})$ is the standard generator. If d divides a, then $\mathcal{D}_{h,a} = \frac{1}{2}H_{m,\mu}$. One denotes by $\mathrm{Pic}^H_\mathbb{Q}(\mathcal{F}_{2d})$ the subspace generated by all NL divisors $\mathcal{D}_{h,a}$, or equivalently, Heegner divisors $H_{m,\mu}$. Maulik–Pandharipande conjectured [MP13, Conjecture 3] the equality

$$\mathrm{Pic}^H_\mathbb{Q}(\mathcal{F}_{2d}) = \mathrm{Pic}_\mathbb{Q}(\mathcal{F}_{2d}).$$

This is now a theorem:

Theorem 2.1 (Theorem 1.8 in [BLMM17])**.** *Let Λ be an even lattice of signature $(2,n)$ with $n \geq 3$, and $\Gamma \subset \widetilde{\mathrm{O}}^+(\Lambda)$ a finite index subgroup. Then the rational Picard group of $\mathcal{D}_\Lambda/\Gamma$ is generated by Heegner divisors:*

$$\mathrm{Pic}^H_\mathbb{Q}\left(\mathcal{D}_\Lambda/\Gamma\right) = \mathrm{Pic}_\mathbb{Q}\left(\mathcal{D}_\Lambda/\Gamma\right).$$

Note that the above theorem in particular implies that irreducible components of $H_{m,\mu}$ must be linear combinations of other Heegner divisors. When $\Gamma = \widetilde{\mathrm{O}}^+(\Lambda)$ the relation is explicit and follows from Eichler's criterion [GHS09, Proposition 3.3], [Son23, Proposition 2.15]: *if Λ splits off two copies of the hyperbolic lattice U, then the $\widetilde{\mathrm{SO}}^+(\Lambda)$-orbit of a primitive element $v \in \Lambda^\vee$ is determined by $Q(v) = m$ and $v + \Lambda \in \Lambda^\vee/\Lambda$.* This leads to the following definition (see [Pet15, BM19]). The *primitive Heegner divisor* $P_{\Delta,\delta}$ is the image via the Γ-quotient map $\pi : \mathcal{D}_\Lambda \longrightarrow \mathcal{D}_\Gamma/\Gamma$ of the cycle

$$\sum_{\substack{v \in \delta + \Lambda \text{ primitive} \\ Q(v) = \Delta}} D_v. \tag{3}$$

When Λ splits off two copies of U, and $\Gamma = \widetilde{\mathrm{O}}^+(\Lambda)$, the divisor $P_{\Delta,\delta}$ is irreducible and reduced when $\delta \neq -\delta$ in $D(\Lambda)$ and otherwise has multiplicity two. The relation between

Heegner and primitive Heegner divisors [BM19, Equations (17) and (18)] is:

$$H_{m,\mu} = \sum_{\substack{r\in\mathbb{Z}_{>0}\\ r^2|m}} \sum_{\substack{\delta\in D(\Lambda)\\ r\delta=\mu}} P_{\frac{m}{r^2},\delta} \quad \text{and} \quad P_{\Delta,\delta} = \sum_{\substack{r\in\mathbb{Z}_{>0}\\ r^2|\Delta}} \mu(r) \sum_{\substack{\sigma\in D(\Lambda)\\ r\sigma=\delta}} H_{\frac{\Delta}{r^2},\sigma}, \tag{4}$$

where the $\mu(\cdot)$ in the second equation stands for the Möbius function. Here $r^2 \mid m$ means exactly that there is a class $\delta \in D(\Lambda)$ such that $m/r^2 \in Q(\delta) + \mathbb{Z}$.

As stated in the introduction, our main object of study is the *NL-cone* $\mathrm{Eff}^{NL}\left(\mathcal{D}_\Lambda/\Gamma\right)$ generated by primitive Heegner divisors, or equivalently irreducible components of Noether-Lefshetz divisors.

2.2. **Rational Picard group of orthogonal modular varieties.** A recently established key feature of our setting is that for a large collection of orthogonal modular varieties the $\mathbb{Q}$-vector space $\mathrm{Pic}_{\mathbb{Q}}\left(\Lambda/\Gamma\right)$ can be seen as a space of vector-valued modular forms. This is what we explain now.

Let Λ be an even lattice of signature $(2, n)$ with quadratic form Q. The discriminant group $D(\Lambda) = \Lambda^\vee/\Lambda$ is a finite abelian group endowed with an induced $\mathbb{Q}/\mathbb{Z}$-valued quadratic form. The group algebra $\mathbb{C}\left[D(\Lambda)\right]$ is finitely generated and we denote the standard generators by $\{\mathfrak{e}_\mu \mid \mu \in D(\Lambda)\}$. The *metaplectic group* $\mathrm{Mp}_2(\mathbb{Z})$ is a double cover of $\mathrm{SL}_2(\mathbb{Z})$ defined as the group of pairs $(A, \phi(\tau))$ where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, and $\phi(\tau)$ is a choice of a square root of the function $c\tau + d$ on the upper half plane $\mathbb{H}$. The product in $\mathrm{Mp}_2(\mathbb{Z})$ is given by $(A_1, \phi_1(\tau)) \cdot (A_2, \phi_2(\tau)) = (A_1A_2, \phi_1(A_2\tau)\phi_2(\tau))$. There is a canonical representation of $\mathrm{Mp}_2(\mathbb{Z})$ attached to Λ called the *Weil representation* $\rho_\Lambda : \mathrm{Mp}_2(\mathbb{Z}) \longrightarrow \mathrm{GL}\left(\mathbb{C}\left[D(\Lambda)\right]\right)$. See [Bor98, Section 4] for a concrete description in terms of the standard generators of $\mathrm{Mp}_2(\mathbb{Z})$. Let $k \in \frac{1}{2}\mathbb{Z}$. A holomorphic function

$$f : \mathbb{H} \longrightarrow \mathbb{C}\left[D(\Lambda)\right]$$

is called a *modular form of weight k and type* ρ_Λ if for all $g = (A, \phi) \in \mathrm{Mp}_2(\mathbb{Z})$ and $\tau \in \mathbb{H}$

$$f(A\tau) = \phi(\tau)^{2k}\rho_\Lambda(g) \cdot f(\tau).$$

and f is holomorphic at the cusp at ∞. Modular forms of weight k and type ρ_Λ form a finite-dimensional $\mathbb{C}$-vector space denoted $\mathrm{Mod}_{k,\Lambda}$. Such a modular form f admits a Fourier expansion centered at the cusp at infinity of the form

$$f = \sum_{\mu\in D(\Lambda)} \sum_{m\in\frac{1}{N}\mathbb{Z}_{\geq 0}} a_{m,\mu} q^m \mathfrak{e}_\mu,$$

where as usual $q = e^{2\pi i\tau}$. Here N is the *level* of Λ, that is, the smallest positive integer such that $N \cdot Q$ is integral on $\Lambda^\vee$. Further, from [Bor99, Lemma 4.2] and [McG03, Theorem 5.6], one can find a basis for $\mathrm{Mod}_{k,\Lambda}$ where all Fourier coefficients are rational numbers.

The modular form f is called a *cusp form* if $a_{0,\mu} = 0$ for all isotropic elements $\mu \in D(\Lambda)$, i.e, the function $\sum_m a_{m,\mu} q^m$ vanishes at the cusp of $\mathbb{H}$. The function f is called an *almost cusp form* if $a_{m,\mu} = 0$ for all isotropic elements μ except possibly $0 \in D(\Lambda)$ (see for instance [Pet15, Section 3.3]). Cusp forms and almost cusp forms form subspaces

$$\mathrm{S}_{k,\Lambda} \subset \mathrm{Mod}^\circ_{k,\Lambda} \subset \mathrm{Mod}_{k,\Lambda}.$$

Let $\widetilde{\Gamma}_\infty$ be the stabilizer in $\mathrm{Mp}_2(\mathbb{Z})$ of the cusp at infinity. Assume further that $2k \equiv 2-n \mod 4$. Then for every half integer $k > 2$ the Eisenstein series

$$E_{k,\Lambda}(\tau) = \sum_{(A,\phi)\in\widetilde{\Gamma}_\infty\backslash \mathrm{Mp}_2(\mathbb{Z})} \phi(\tau)^{2k} \cdot \rho_\Lambda(A,\phi)^{-1}\mathfrak{e}_0 = \sum_{m,\mu} e_{m,\mu} q^m \mathfrak{e}_\mu \tag{5}$$

is in $\mathrm{Mod}_{k,\Lambda}$. The coefficients $e_{m,\mu}$ are rational numbers that were computed in [BK01]. As $\mathbb{Q}$-vector spaces one has

$$\mathrm{Mod}^\circ_{k,\Lambda} = \mathbb{Q}E_{k,\Lambda} \oplus S_{k,\Lambda}.$$

Following the notation in [Pet15, BM19], consider the coefficient extraction functionals

$$\begin{array}{rcl} c_{m,\mu} : \mathrm{Mod}^\circ_{k,\Lambda} & \longrightarrow & \mathbb{Q} \\ f & \mapsto & c_{m,\mu}(f). \end{array}$$

where $c_{m,\mu}(f)$ is the (m,μ)-th Fourier coefficient $a_{m,\mu}$ of f. These functionals generate $\left(\mathrm{Mod}^\circ_{k,\Lambda}\right)^\vee$. The key theorem that allows us to study the effective cone is the following:

Theorem 2.2 ([Bor99, McG03, Bru02a, Bru14, BLMM17])**.** *Let Λ be an even lattice of signature $(2,n)$ with $n \geq 3$ splitting off two copies of U, and $\Gamma \subset \widetilde{\mathrm{O}}^+(\Lambda)$ a finite index subgroup. Then the map*

$$\varphi\colon \left(\mathrm{Mod}^\circ_{k,\Lambda}\right)^\vee \longrightarrow \mathrm{Pic}_{\mathbb{Q}}\left(\mathcal{D}_\Lambda/\Gamma\right), \quad c_{m,\mu} \mapsto H_{-m,\mu} \tag{6}$$

is an isomorphism of $\mathbb{Q}$-vector spaces for $k = 1 + n/2$.

Remark 2.3. Under the above isomorphism φ, the Hodge class λ is identified with the functional $-c_{0,0}$ sending $E_{k,\Lambda}$ to -1 and $S_{k,\Lambda}$ to 0.

The fact that φ is a well-defined $\mathbb{Q}$-homomorphism follows from [Bor99, McG03], injectivity follows from [Bru02a, Theorem 0.4] and [Bru14, Theorem 1.2], and surjectivity is Theorem 2.1.

2.3. **Effective and NL cones.** It was shown in [BM19] that, on the right-hand side of (6), the functionals $c_{m,\mu}$ converge projectively to $-c_{0,0}$ as m grows. This implies that the cone spanned by all $H_{m,\mu}$ is polyhedral. Using the formula (4), Bruinier–Möller moreover show that the cone $\mathrm{Eff}^{NL}\left(\mathcal{D}_\Lambda/\widetilde{\mathrm{O}}^+(\Lambda)\right)$ generated by primitive Heegner divisors $P_{\Delta,\delta}$ is polyhedral, answering [Pet15, Question 4.5.2]. More precisely, [BM19] shows that there is a neighborhood $\mathcal{U}$ of $\mathbb{Q}_{\geq 0}\lambda$ strictly contained in $\mathrm{Eff}^{NL}\left(\mathcal{D}_\Lambda/\widetilde{\mathrm{O}}^+(\Lambda)\right)$ and a value Δ_0

such that for all $\Delta \geq \Delta_0$, we have $P_{\Delta,\delta} \in \mathcal{U}$. The NL-cone $\mathrm{Eff}^{NL}\left(\mathcal{D}_\Lambda/\widetilde{\mathrm{O}}^+(\Lambda)\right)$ is then the convex hull of the divisors $P_{\Delta,\delta}$ for $\Delta \leq \Delta_0$.

Formulas for the NL cones $\mathrm{Eff}^{NL}(\mathcal{F}_{2d})$ for low values of d were conjectured in [Pet15] by looking at truncated Fourier coefficients of the modular forms generating $\mathrm{Mod}^\circ_{\frac{21}{2},\Lambda_{2d}}$, see [Pet15, Remark 4.7.1]. More precisely, for $d \leq 18$ Peterson used (6) to compute the cone generated by the $8d$ generators $P_{Q(\delta)+j,\delta}$ for $\delta \in D(\Lambda_{2d})$ and $j \in \{0,1,2,3\}$. He then conjectured that this cone coincides with $\mathrm{Eff}^{NL}(\mathcal{F}_{2d})$ for these values of d.

Confirming these formulas for a given d requires explicitly computing the $\mathcal{U}$ and Δ_0 described above. This has to do with finding concrete bounds analogous to *Deligne's bound* for scalar-valued Hecke eigenforms of integral weight. Once these $\mathcal{U}$ and m_0 are computed, calculating $\mathrm{Eff}^{NL}\left(\mathcal{D}_\Lambda/\widetilde{\mathrm{O}}^+(\Lambda)\right)$ can be accomplished by computer. See Section 3 for more details.

Let X be a normal $\mathbb{Q}$-factorial quasi-projective variety with $\mathrm{Pic}_{\mathbb{Q}}(X)$ a finite dimensional $\mathbb{Q}$-vector space. The *effective cone* $\mathrm{Eff}(X)$ is the cone in $\mathrm{Pic}_{\mathbb{Q}}(X)$ generated by all effective $\mathbb{Q}$-divisors up to linear equivalence:

$$\mathrm{Eff}(X) = \langle E \in \mathrm{Pic}_{\mathbb{Q}}(X) \,|\, E \text{ is effective} \rangle_{\mathbb{Q}_{\geq 0}}.$$

When X is projective and $h^1(X,\mathcal{O}_X) = 0$, then $\mathrm{Pic}_{\mathbb{Q}}(X)$ coincides with the Neron-Severi group $\mathrm{NS}(X)_{\mathbb{Q}}$ and one recovers the standard definition. The definition for $\mathbb{R}$-divisors is the same. Further, the cone is often not closed and the closure is called the *pseudo-effective cone*, denoted $\overline{\mathrm{Eff}}(X)$.

3. NL-cone computations

Throughout this section, we assume that Λ is a lattice of signature $(2,n)$ with $n \geq 3$ splitting off one copy of the hyperbolic plane. We moreover consider the half-integer $k = 1 + n/2$.

As described in Section 2.3, in order to compute the NL-cone $\mathrm{Eff}^{NL}\left(\mathcal{D}_\Lambda/\widetilde{\mathrm{O}}^+(\Lambda)\right)$ for a given lattice Λ, one needs to calculate a neighborhood $\mathcal{U}$ of $\mathbb{Q}_{\geq 0}\lambda$ strictly contained in $\mathrm{Eff}^{NL}\left(\mathcal{D}_\Lambda/\widetilde{\mathrm{O}}^+(\Lambda)\right)$ and an explicit value Ω such that $P_{\Delta,\delta} \in \mathcal{U}$ for all $\Delta > \Omega$. Further, as in Subsection 2.2, we view $\mathrm{Mod}_{k,\Lambda}$ and $S_{k,\Lambda}$ as $\mathbb{Q}$-vector spaces.

In order to find such an explicit Ω, we fix a rational basis $\{f_1, \dots f_M\}$ for $S_{k,\Lambda}$. Let $e = E_{k,\Lambda}$ be the Eisenstein series defined in (5). We use the isomorphism (6) to identify each $H_{-m,\mu}$ with the coefficient functional $c_{m,\mu}$ and hence a tuple

$$c_{m,\mu}(e, f_1, \dots, f_M) = (c_{m,\mu}(e), c_{m,\mu}(f_1), \dots, c_{m,\mu}(f_M)) \in \mathbb{Q}^{M+1}.$$

Intuitively, as m increases, the coefficients $c_{m,\mu}(e)$ of $E_{k,\Lambda}$ grow more rapidly than those of any cusp form, and therefore $c_{m,\mu}(e, f_1, \dots, f_M)$ converges projectively to $(-1, 0, \dots, 0)$, which corresponds to the Hodge class λ (see Remark 2.3). This convergence is proved in [BM19, Proposition 4.5]. However to produce the required neighborhood $\mathcal{U}$ and bound

Ω, we need to make this convergence quantitative: we need explicit upper bounds for the Fourier coefficients of vector-valued cusp forms and an explicit lower bound for the coefficients of the Eisenstein series.

The coefficients of e can be expressed in closed form [BK01] and a lower bound of the form $c_{m,\mu}(e) \geq C_{k,\Lambda} \cdot m^{k-1}$, where $C_{k,\Lambda}$ is an explicit positive constant depending only on the lattice Λ and weight k, easily follows, cf. [BM19, Propositions 3.2 and 4.5]. As for cusp forms, despite the considerable literature on bounds for the growth of Fourier coefficients, we are unaware of a general bound (with explicit constants) that applies to our situation so we derive one below. The bound we derive is only the trivial bound $\mathrm{O}(m^{k/2})$, but this is sufficient to distinguish it from the growth of the lower bound for $c_{m,\mu}(e)$.

We will use the fact that the space of cusp forms $S_{k,\Lambda}$ is spanned by Poincaré series

$$P_{k,(m,\mu)}(\tau) = \frac{1}{2} \sum_{\substack{c,d\in\mathbb{Z},\\ \gcd(c,d)=1}} (c\tau+d)^{-k} e^{2\pi i m \frac{a\tau+b}{c\tau+d}} \rho_\Lambda\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \mathfrak{e}_\mu.$$

These are characterized through the Petersson inner product

$$\langle f, g\rangle := \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} \sum_{\mu\in D(\Lambda)} f_\mu(\tau)\overline{g_\mu(\tau)} y^k \frac{\mathrm{d}x\,\mathrm{d}y}{y^2}, \quad f, g \in S_{k,\Lambda}$$

by the fact that they represent (up to a constant factor) the coefficient extraction functionals: an arbitrary cusp form

$$f(\tau) = \sum_{\mu\in D(\Lambda)} \sum_{m\in\frac{1}{N}\mathbb{Z}_{>0}} a_{m,\mu} q^m \mathfrak{e}_\mu \tag{7}$$

has Fourier coefficients $a_{m,\mu}$ which can be written

$$a_{m,\mu} = \frac{(4\pi m)^{k-1}}{\Gamma(k-1)} \left\langle f, P_{k,(m,\mu)}\right\rangle. \tag{8}$$

This implies that to bound the coefficients of arbitrary cusp forms, it is sufficient to bound the growth of the “diagonal” coefficients of Poincaré series. More precisely:

Lemma 3.1. *Suppose the coefficients of*

$$P_{k,(m,\mu)}(\tau) = \sum_{\beta\in D(\Lambda)} \sum_{n\in Q(\beta)+\mathbb{Z}} c_{m,\gamma}(n,\beta) q^n \mathfrak{e}_\beta$$

satisfy a bound of the form

$$|c_{m,\mu}(m,\mu)| \leq C \cdot m^A$$

for some positive constants A and C. Then the coefficients of every cusp form (7) *satisfy the bound*

$$|a_{m,\mu}| \leq \tilde{C} \cdot m^{A/2+(k-1)/2} \cdot \|f\|$$

with constant

$$\tilde{C} := \frac{(4\pi)^{(k-1)/2}}{\sqrt{\Gamma(k-1)}} \cdot \sqrt{C}.$$

Proof. From (8) it follows that the Petersson norm of $P_{k,(m,\mu)}$ is

$$\begin{aligned}\|P_{k,(m,\mu)}\| = \sqrt{\langle P_{k,(m,\mu)}, P_{k,(m,\mu)}\rangle} &= \frac{\sqrt{\Gamma(k-1)}}{(4\pi m)^{(k-1)/2}} \cdot |c_{m,\mu}(m,\mu)|^{1/2} \\ &\leq \sqrt{C \cdot \Gamma(k-1)}(4\pi)^{k/2-1/2} \cdot m^{A/2+(1-k)/2}.\end{aligned}$$

The Cauchy–Schwarz inequality then yields

$$\begin{aligned}|a_{m,\mu}| &= \frac{(4\pi m)^{k-1}}{\Gamma(k-1)} |\langle f, P_{k,(m,\mu)}\rangle| \\ &\leq \frac{(4\pi m)^{k-1}}{\Gamma(k-1)} \cdot \|f\| \cdot \|P_{k,(m,\mu)}\| \\ &\leq \frac{(4\pi)^{(k-1)/2}\sqrt{C}}{\sqrt{\Gamma(k-1)}} m^{A/2+(k-1)/2} \cdot \|f\|.\end{aligned}$$

□

The following lemma gives an explicit bound of the form required in Lemma 3.1.

Lemma 3.2. *For any half-integer $k \geq 5/2$, the diagonal coefficients $c_{m,\mu}(m,\mu)$ of $P_{k,(m,\mu)}$ satisfy*

$$|c_{m,\mu}(m,\mu)| \leq C \cdot m$$

with constant

$$C = C(k) = \frac{(2\pi)^k}{\Gamma(k) \cdot (k-2)} + 2.125.$$

Proof. From [Bru02a, Chapter 1.2], the Fourier coefficients of

$$P_{k,(m,\mu)} = \sum_{\beta \in D(\Lambda)} \sum_{n \in Q(\beta)+\mathbb{Z}} c_{m,\mu}(n,\beta) q^n \mathfrak{e}_\beta$$

are given by the formula

$$c_{m,\mu}(n,\beta) = 2\pi \Big(\frac{m}{n}\Big)^{(1-k)/2} \sum_{c=1}^{\infty} \frac{1}{c} J_{k-1}(4\pi\sqrt{mn}/c) \cdot \mathrm{Re}\Big[e^{-\pi i k} K_c(\mu,m,\beta,n)\Big],$$

where K_c is the generalized Kloosterman sum

$$K_c(\mu,m,\beta,n) = \sum_{d \in (\mathbb{Z}/c\mathbb{Z})^\times} e^{2\pi i (ma+nd)/c} \langle \rho(M)^{-1}\mathfrak{e}_\mu, \mathfrak{e}_\beta\rangle,$$

and J is the usual Bessel function. For our application, the trivial bound $|K_c(\mu,m,\beta,n)| \leq c$ will be enough.

The Bessel function satisfies the bounds

$$|J_{k-1}(x)| \leq \frac{M}{x^{1/3}}, \quad \text{where } M \approx 0.78574687$$

(cf. [Lan00]) and

$$|J_{k-1}(x)| \leq \frac{x^{k-1}}{2^{k-1}\Gamma(k)}$$

(cf. [NIST:DLMF], 10.14.4). For small values of c (say $c \leq n$), we use the first bound:

$$\begin{aligned}\Big|\sum_{c=1}^{n} \frac{1}{c} J_{k-1}(4\pi\sqrt{mn}/c)\cdot \mathrm{Re}\Big[e^{-\pi ik}K_c(\mu,m,\beta,n)\Big]\Big| &\leq (4\pi\sqrt{mn})^{-1/3}M\cdot\sum_{c=1}^{n} c^{1/3} \\ &\leq (4\pi)^{-1/3}m^{-1/6}M\cdot n^{7/6}.\end{aligned}$$

We use the second bound for $c > n$:

$$\begin{aligned}\Big|\sum_{c>n} \frac{1}{c} J_{k-1}(4\pi\sqrt{mn}/c)\cdot \mathrm{Re}\Big[e^{-\pi ik}K_c(\mu,m,\beta,n)\Big]\Big| &\leq \frac{(2\pi)^{k-1}(mn)^{(k-1)/2}}{\Gamma(k)}\sum_{c>n}\frac{1}{c^{k-1}} \\ &\leq \frac{(2\pi)^{k-1}m^{(k-1)/2}n^{(3-k)/2}}{\Gamma(k)(k-2)}.\end{aligned}$$

(In the last step, we used $\sum_{c>n} c^{1-k} < \int_n^\infty \frac{\mathrm{d}t}{t^{k-1}} = \frac{n^{2-k}}{k-2}$.) Altogether, we have

$$|c_{m,\mu}(n,\beta)| \leq 2\pi\Big(\frac{m}{n}\Big)^{(1-k)/2}\cdot(4\pi)^{-1/3}m^{-1/6}M\cdot n^{7/6} + \frac{(2\pi)^k}{\Gamma(k)(k-2)}n.$$

For the diagonal coefficient $(m,\mu = (n,\beta)$, we obtain

$$|c_{m,\mu}(m,\mu)| \leq 2^{1/3}\pi^{2/3}M\cdot m + \frac{(2\pi)^k}{\Gamma(k)(k-2)}\cdot m.$$

The claim follows because $2^{1/3}\pi^{2/3}M < 2.125$. □

We now describe how to use the bounds of Lemma 3.1 to make the argument of [BM19] explicit, thereby proving Theorem 1.1.

We will first describe how to compute the cone of Heegner divisors $\mathrm{Eff}^H\Big(\mathcal{D}_\Lambda/\widetilde{\mathrm{O}}^+(\Lambda)\Big)$. Let $\big(\mathrm{Mod}^\circ_{k,\Lambda}\big)^\vee$ be the space of linear functionals on $\mathrm{Mod}^\circ_{k,\Lambda}$ and consider the cone $\mathcal{C}$ generated by the coefficient extraction functionals

$$c_{m,\mu} : \mathrm{Mod}^\circ_{k,\Lambda} \longrightarrow \mathbb{Q}, \quad \sum a_{m,\mu}q^m\mathfrak{e}_\mu \mapsto a_{m,\mu}.$$

Write $c_{m,\mu} = \gamma_{m,\mu}e + s_{m,\mu}$, where e is the functional

$$e(E_{k,\Lambda}) = -1, \quad e\Big|_{S_{k,L}} = 0,$$

and $s_{m,\mu}(E_{k,\Lambda}) = 0$. In particular,

$$E_{k,\Lambda}(\tau) = \mathfrak{e}_0 - \sum_{m,\mu}\gamma_{m,\mu}q^m\mathfrak{e}_\mu.$$

We need to find an open neighborhood of e contained in the cone $\mathcal{C}$. As in [BM19], there is a finite set of indices (m_i,μ_i), $1 \leq i \leq N$ and positive rationals λ_i such that

c_{m_i,μ_i} spans $S^\vee_{k,L}$ and

$$\sum_{i=1}^{N} \lambda_i c_{m_i,\mu_i} = e. \tag{9}$$

Following [BM19, Proposition 3.3], the λ_i can be constructed as follows. For b sufficiently large (explicit) positive integer let f be the weakly holomorphic modular form

$$f(\tau) = \Delta(\tau)^{-b} \cdot E_{(2-k)+12b,\Lambda(-1)}(\tau), \text{ where } \Delta(\tau) = \eta(\tau)^{24} = q \cdot \prod_{n\geq 1}(1-q^n)^{24}$$

is the scalar-valued discriminant modular form. Recall that

$$\Delta(\tau)^{-b} = q^{-b} \cdot \left(\prod_{n\geq 1} \frac{1}{1-q^n}\right)^{24b} = q^{-b} \cdot \left(\sum_{n=0}^{\infty} p(n) q^n\right)^{24b},$$

where $p(n)$ is the number of partitions of n. In particular the coefficient of q^m in the expansion of Δ^{-b} is zero for $m < -b$ and the Fourier coefficients of the product $f(\tau)$ can be computed explicitly. We write

$$f(\tau) = \sum_{\mu \in D(\Lambda(-1))} \sum_{m \in \mathbb{Q}} \alpha_{m,\mu} q^m \mathfrak{e}_\mu.$$

As a consequence of the residue theorem one has that for any cusp form $f \in S_{k,\Lambda}$,

$$\sum_{\substack{(m,\mu)\\ -b\leq m<0}} \alpha_{m,\mu} c_{-m,\mu}(f) = 0$$

and we simply have to choose $b \geq \lceil k/12 \rceil$ large enough such that the above functionals $c_{-m,\mu}$ span $S^\vee_{k,\Lambda}$. Then taking such a collection as a generating set and $\lambda_i = \frac{\alpha_{-m_i,\mu_i}}{\alpha_{0,0}}$ with $m_i > 0$ one can ensure (9) holds. This is the only input needed to produce a bound for a generating set of both the Heegner and the NL-cones.

Example 3.3. As an example, we take the lattice $\Lambda = \Lambda_4$ corresponding to the moduli of degree four K3 surfaces. Then $S_{\frac{21}{2},\Lambda_4}$ is two dimensional generated by

$$\begin{aligned} f_1 = &\left(-128q - 57344q^2 + \dots\right)\mathfrak{e}_0 + \left(q^{1/8} - 7q^{9/8} + \dots\right)\mathfrak{e}_{\ell_*} \\ &+ \left(4864q^{3/2} + 368640q^{5/2} + \dots\right)\mathfrak{e}_{2\ell_*} + \left(q^{1/8} - 7q^{9/8} + \dots\right)\mathfrak{e}_{3\ell_*}, \\ f_2 = &\left(-14q - 568q^2 + \dots\right)\mathfrak{e}_0 + \left(32q^{9/8} + 544q^{17/8} + \dots\right)\mathfrak{e}_{\ell_*} \\ &+ \left(q^{1/2} - 188q^{3/2} + \dots\right)\mathfrak{e}_{2\ell_*} + \left(32q^{9/8} + 544q^{17/8} + \dots\right)\mathfrak{e}_{3\ell_*}. \end{aligned}$$

Here the dots mean higher-order terms. Since $\lceil k/12 \rceil = 1$, we take $b = 1$. Then

$$\begin{aligned} \Delta^{-1}(\tau) &= q^{-1}\left(1 + 1q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + \dots\right)^{24} \\ &= q^{-1} + 24 + 324q + 3128q^2 + \dots \end{aligned}$$

and one obtains

$$\Delta^{-1} \cdot E_{\frac{7}{2},\Lambda(-1)} = q^{-1}\mathfrak{e}_0 + 64q^{-1/8}\mathfrak{e}_{\ell_*} + 14q^{-1/2}\mathfrak{e}_{2\ell_*} + 64q^{-1/8}\mathfrak{e}_{3\ell_*} + 84\mathfrak{e}_0 + \sum_{\substack{(m,\mu)\\ m>0}} \alpha_{m,\mu} q^m \mathfrak{e}_\mu.$$

Recall that $c_{m,\mu} = c_{m,-\mu}$. One easily checks that the set of all $c_{m,\mu}$ with $0 < m \leq 1$, in this case $\{c_{m_i,\mu_i}\}_{i=1}^4$ with indices

$$(m_1, \mu_1) = (1, 0), (m_2, \mu_2) = (1/8, \ell_*), (m_3, \mu_3) = (1/2, 2\ell_*), \text{and} (m_4, \mu_4) = (1/8, 3\ell_*)$$

generates $S^\vee_{k,\Lambda_4}$. Then with $\lambda_1 = \frac{1}{84}$, $\lambda_2 = \lambda_4 = \frac{64}{84}$, and $\lambda_3 = \frac{14}{84}$, Equation (9) holds.

We will need to transfer these results for the Petersson norm in terms of the ℓ^2-norm on $\mathbb{Q}^M$. Recall that we identify each functional $s_{m,\mu}$ with the tuple

$$(s_{m,\mu}(f_1), ..., s_{m,\mu}(f_M)) \in \mathbb{Q}^M$$

where $f_1, ..., f_M$ is a rational basis of $S_{k,\Lambda}$.

Define an inner product on $\mathrm{Mod}^\circ_{k,\Lambda}$ as follows: for $f \in S_{k,\Lambda}$ then $\|f\|$ is the usual Petersson norm and we declare the Eisenstein series $E_{k,L}$ to have norm one and be orthogonal to $S_{k,\Lambda}$.

To pass from $\|f\|$ to the ℓ^2-norm $\|f\|_{\ell^2}$, we need a rational basis whose Petersson norms can be estimated explicitly. One such basis was described in [Wil18]:

$$f_{m,\mu} := \sum_{\lambda=1}^{\infty} P_{k,(\lambda^2 m, \lambda\mu)}. \tag{10}$$

These are convenient because their Petersson norm is easy to bound using Lemma 3.1. Indeed, writing $f_{m,\mu} = \sum c(n,\gamma) q^n \mathfrak{e}_\gamma$, one has

$$\begin{aligned}
\|f_{m,\mu}\|^2 &\leq \sum_{\lambda=1}^{\infty} \frac{\Gamma(k-1)}{(4\pi\lambda^2 m)^{k-1}} |c(\lambda^2 m, \lambda\mu)| \\
&\leq \frac{\tilde{C} \cdot \Gamma(k-1) \cdot \|f_{m,\mu}\|}{(4\pi)^{k-1}} \sum_{\lambda=1}^{\infty} \frac{(\lambda^2 m)^{k/2}}{(\lambda^2 m)^{k-1}} \\
&= \frac{\tilde{C} \cdot \Gamma(k-1) \cdot \zeta(k-2)}{(4\pi)^{k-1}} \cdot \|f_{m,\mu}\| \cdot m^{1-k/2}.
\end{aligned}$$

Therefore,

$$\|f_{m,\mu}\| \leq \frac{\tilde{C} \cdot \Gamma(k-1) \cdot \zeta(k-2)}{(4\pi)^{k-1}} \cdot m^{1-k/2}.$$

So with respect to this basis, the Petersson norm and the ℓ^2-norm on $\mathbb{Q}^M$ of $s_{m,\mu}$ are related by

$$
\begin{aligned}
\|s_{m,\mu}\| &= \sup_{f\neq 0} \frac{|s_{m,\mu}(f)|}{\|f\|} \\
&\geq \frac{1}{\max_i \|f_i\|}\sqrt{\frac{1}{M}\sum_{i=1}^{M} |s_{m,\mu}(f_i)|^2} \qquad (11)\\
&\geq \frac{(4\pi)^{k-1}\cdot \max_i m_i^{k/2-1}}{\tilde{C}\cdot \Gamma(k-1)\zeta(k-2)\sqrt{M}} \cdot \|s_{m,\mu}\|_{\ell^2}.
\end{aligned}
$$

Now we can bound the number of generators of the cone $\mathcal{C}$.

Theorem 3.4. *For any choice of λ_i and m_i as above (see Equation (9)), the cone $\mathcal{C}$ generated by all coefficient functionals is already generated by $c_{m,\mu}$ with*

$$m \leq \Big(\frac{R\cdot C_{k,\Lambda}}{B}\Big)^{2/(2-k)},$$

where $C_{k,\Lambda}$ is any constant such that the Fourier coefficients $e(m,\mu)$ of $E_{k,\Lambda}$ are bounded from below by

$$|e_{m,\mu}| \geq C_{k,\Lambda}\cdot m^{k-1},$$

where $R > 0$ is such that the convex hull $\mathcal{C}_S$ of $\frac{s_{m_i,\mu_i}}{\gamma_{m_i,\mu_i}}$ contains the ball of radius R with respect to the ℓ^2-norm, and where

$$B := \frac{(\tilde{C})^2\Gamma(k-1)\zeta(k-2)\sqrt{M}}{(4\pi)^{k-1}\cdot \max_i m_i^{k/2-1}}$$

where $\tilde{C}$ is the constant from Lemma 3.1.

Remark 3.5. Note that $\mathcal{C}_S$ contains an open neighborhood of 0 by [BM19]. To compute a concrete radius R, we write $\mathcal{C}_S \subset \mathbb{Q}^M$ as an intersection of finitely many half-planes, say $\{x : \langle v,x\rangle \leq a\}$, and take R to be the minimum of $|a|/\|v\|_{\ell^2}$, where the latter is the standard ℓ^2-norm on $\mathbb{Q}^M$. As for the choice of a constant $C_{k,\Lambda}$, when the discriminant of Λ is D, it that can be derived from [BM19] is

$$C_{k,\Lambda} = \frac{16}{5}\Big(\frac{\pi}{2}\Big)^k \cdot \frac{\sqrt{D}}{\zeta(k-1/2)\Gamma(k)} \prod_{\substack{\text{primes}\\ p|D}} \frac{1-1/p}{1-1/p^{2k-1}}.$$

As an example, for the lattices $\Lambda = \Lambda_d$ and $k = 21/2$, this bound is approximately

$$C_{k,\Lambda} \approx 0.0002286\cdot \sqrt{d} \prod_{\substack{p|d\\ p\,\text{odd}}} \frac{1-1/p}{1-1/p^{20}}.$$

Proof of Theorem 3.4. The coefficient functional $s_{m,\mu}$ is bounded in operator norm by

$$\|s_{m,\mu}\| \leq \tilde{C}\cdot m^{k/2}$$

by Lemma 3.1, and therefore in ℓ^2-norm by

$$\|s_{m,\mu}\|_{\ell^2} \leq B \cdot m^{k/2}$$

with the constant B by (11). Recall that $\|e\| = 1$. Since $C_{k,L}$ is such that

$$\gamma_{m,\mu} \geq C_{k,\Lambda} \cdot m^{k-1},$$

we have

$$\left\| \frac{c_{m,\mu}}{\gamma_{m,\mu}} - e \right\|_{\ell^2} \leq \frac{B}{C_{k,\Lambda}} \cdot m^{1-k/2}.$$

Therefore, if $\frac{B}{C_{k,\Lambda}} m^{1-k/2} < R$ then $c_{m,\mu}$ belongs to the interior of $\mathcal{C}$.

□

Since the functionals $c_{m,\mu}$ correspond to the (non-primitive) Heegner divisors $H_{m,\mu}$ under the isomorphism (6) of Theorem 2.2, Theorem 3.4 describes a generating set for the Heegner cone $\mathrm{Eff}^H\left(\mathcal{D}_\Lambda/\widetilde{\mathrm{O}}^+(\Lambda)\right)$.

We will now use the bounds of Theorem 3.4 in order to compute the NL-cone $\mathrm{Eff}^{NL}\left(\mathcal{D}_\Lambda/\widetilde{\mathrm{O}}^+(\Lambda)\right)$. As is the case in [BM19], we need to impose the added assumption that Λ splits off **two** copies of the hyperbolic plane.

To state the explicit bound Ω in the case of the $P_{\Delta,\delta}$ generating $\mathrm{Eff}^{NL}\left(\mathcal{D}_\Lambda/\widetilde{\mathrm{O}}^+(\Lambda)\right)$, define the functionals

$$p_{\Delta,\delta} := \sum_{\substack{r \in \mathbb{Z}_{>0} \\ r^2 | \Delta}} \mu(r) \sum_{\substack{\sigma \in D(\Lambda) \\ r\sigma = \delta}} c_{\Delta/r^2,\sigma},$$

such that via the isomorphism (6) one has $\varphi(p_{\Delta,\delta}) = P_{\Delta,\delta}$ is the corresponding primitive Heegner divisor by Equation 4. Let $\mathcal{P}$ be the cone generated by the $p_{\Delta,\delta}$. As in the case of the Heegner cone, using the isomorphism of Theorem 2.2, a description of the generators of $\mathcal{P}$ gives a description of the generators of $\mathrm{Eff}^{NL}\left(\mathcal{D}_\Lambda/\widetilde{\mathrm{O}}^+(\Lambda)\right)$.

Theorem 3.6. *Let $B, C_{k,\Lambda}$ and R be the constants of Theorem 3.4 and assume Λ has discriminant D and splits off two copies of the hyperbolic plane. The cone $\mathcal{P}$ is already generated by $p_{\Delta,\delta}$ with*

$$\Delta \leq \left(\frac{R \cdot C_{k,\Lambda} \cdot M}{B \cdot (1 + D \cdot (\zeta(k) - 1))^2} \right)^{2/(2-k)},$$

where

$$M := 1 - \frac{1}{2}\Big(\prod_{p\, prime} \left(1 + \frac{1}{p(p-1)}\right) - \prod_{p\, prime} \left(1 - \frac{1}{p(p-1)}\right)\Big) > 0.215.$$

Proof. By Lemma 3.1 and the triangle inequality, for any cusp form f, we have

$$\begin{aligned}|p_{\Delta,\delta}(f)| &\leq \tilde{C}\cdot\|f\|\cdot\sum_{\substack{r\in\mathbb{Z}_{>0}\\ r^2|\Delta}}\sum_{\substack{\sigma\in D(\Lambda)\\ r\sigma=\delta}}\Big(\frac{\Delta}{r^2}\Big)^{k/2}\\ &\leq \tilde{C}\cdot\|f\|\cdot\Delta^{k/2}\cdot\sum_{r=1}^{\infty}r^{-k}\cdot|\{\sigma\in D(\Lambda):\ r\sigma=0\}|\\ &\leq \tilde{C}\cdot\Delta^{k/2}\cdot\|f\|\cdot\Big(1+D\cdot(\zeta(k)-1)\Big).\end{aligned}$$

On the other hand, if $E_{k,\Lambda}$ denotes the Eisenstein series then the proof of [BM19, Proposition 4.5] shows that

$$|p_{\Delta,\delta}(E_{k,\Lambda})|\geq|c_{\Delta,\delta}(E_{k,\Lambda})|\cdot M$$

with the constant M defined above.

So we can copy the proof of Theorem 3.4, with the upper and lower bounds for $c_{m,\mu}$ replaced by those for $p_{\Delta,\delta}$: we multiply $C_{k,\Lambda}$ by M and $\tilde{C}$ (as part of the constant B) by $\zeta(k)\cdot D$. □

Example 3.7. Continuing Example 3.3, the special basis (10) for $S_{\frac{21}{2},\Lambda_4}$ consists of the series

$$\begin{aligned}f_{1/8,\ell_*}&=\frac{7159053}{14318102}f_1+\frac{7683852}{7159051}f_2\\ f_{1/2,2\ell_*}&=\frac{1}{7159051}f_1+\frac{209563208}{221930581}f_2.\end{aligned}$$

With respect to this basis, the convex set $\mathcal{C}_S$ is the triangle with vertices

$$(-7159053/4,-1/2),(-1/2,-3880799/602547),(2143005/2873041,122245370/979706981).$$

This triangle can be described by the inequalities $x\in\mathbb{R}^2$ with

$$\langle(2,602547),x\rangle\geq-3880800,\ \langle(38,-108856407),x\rangle\geq-13582800,$$
$$\langle(-3175198,602547),x\rangle\geq-2293200,$$

so we obtain the radius

$$R=\min\Big(\frac{3880800}{\|(2,602547)\|},\frac{13582800}{\|(38,-108856407)\|},\frac{2293200}{\|(-3175198,602547)\|}\Big)\approx 0.1248$$

for the largest incircle centered at zero.

We have implemented Sage package [Wila], which, given a lattice Λ satisfying the given hypotheses of this section, applies method described above together with the bounds of Theorem 3.6 in order to compute the NL-cone $\mathrm{Eff}^{NL}\Big(\mathcal{D}_\Lambda/\widetilde{\mathrm{O}}^+(\Lambda)\Big)$.

The bounds above are far from being sharp. For example, with $k=21/2$ and $\Lambda=\Lambda_d$, $d\leq 10$, the upper bound for Δ in Theorem 3.7 is given in the following table (rounded to three decimal places):

	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$
Bound	132.245	101.641	110.616	110.252	119.500
	$d=6$	$d=7$	$d=8$	$d=9$	$d=10$
Bound	130.571	119.825	124.493	142.932	140.355

On the other hand, in all cases we were able to compute, the cone of primitive Heegner divisors is already generated in discriminant $\Delta \leq 2$. As a practical matter, we found it far more efficient to compute the cone generated by Heegner divisors with $\Delta \leq 2$ and then check afterwords that it contains all $P_{\Delta,\delta}$ with Δ up to the above bound.

In Sections 5–7 below we explicitly compute $\mathrm{Eff}^{NL}\left(\mathcal{D}_\Lambda\big/\widetilde{\mathrm{O}}^+(\Lambda)\right)$ in the cases of moduli of polarized K3 surfaces and hyperkähler manifolds.

4. Finite maps and NL-cones

Now we place ourselves in the following situation. Let Λ, Λ' be two even lattices of signature $(2,n)$ with $n \geq 3$, and $\Lambda' \subset \Lambda$ a finite index embedding. Let $\Gamma \subset \widetilde{\mathrm{O}}^+(\Lambda)$ be an arithmetic group and Γ' a finite-index subgroup of the stabilizer of Λ' in Γ. Then both Γ and Γ' act on the same period domain $\mathcal{D} = \mathcal{D}_\Lambda = \mathcal{D}_{\Lambda'}$ and the modular projection $\pi_\Gamma : \mathcal{D} \longrightarrow \mathcal{D}\big/\Gamma$ factors

$$\mathcal{D} \xrightarrow{\pi_{\Gamma'}} \mathcal{D}\big/\Gamma' \xrightarrow{\phi} \mathcal{D}\big/\Gamma. \tag{12}$$

Proposition 4.1. *In the above setting,*

$$\phi^*\mathrm{Eff}^{NL}\left(\mathcal{D}\big/\Gamma\right) \subset \mathrm{Eff}^{NL}\left(\mathcal{D}\big/\Gamma'\right) \quad \textit{and} \quad \phi_*\mathrm{Eff}^{NL}\left(\mathcal{D}\big/\Gamma'\right) \subset \mathrm{Eff}^{NL}\left(\mathcal{D}\big/\Gamma\right).$$

Proof. Let P be an irreducible component of $H_{m,\mu}$ in $\mathcal{D}\big/\Gamma$. By definition (2), there exists an hyperplane $D_v \subset \mathcal{D}$ with $v \in \Lambda^\vee \subset \Lambda_\mathbb{Q}$ such that $Q(v) = m, v \equiv \mu \mod \Lambda$ and $\pi_\Gamma(D_v) = P$. Consider the Γ-invariant hyperplane arrangement

$$\bigcup_{g\in\Gamma} D_{g(v)}.$$

Note that $g(v) \in \Lambda^\vee \subset (\Lambda')^\vee$. In particular each $\pi_{\Gamma'}(D_{g(v)})$ is a component of $H_{m,\delta}$ with $g(v) \equiv \delta \mod \Lambda'$ and up to positive multiplicities

$$\phi^*P \leq \pi_{\Gamma'}\left(\bigcup_{g\in\Gamma} D_{g(v)}\right) \leq \sum_{\delta\in D(\Lambda')} H_{m,\delta}.$$

This shows that ϕ^*P is a sum of irreducible components of Heegner $\mathbb{Q}$-divisors on $\mathcal{D}\big/\Gamma'$. Similarly, let P be an irreducible component of a Heegner divisor $H_{m,\mu}$ in $\mathcal{D}\big/\Gamma'$. There exists $v \in (\Lambda')^\vee$ with $\sum_{g\in\Gamma'} D_{g(v)}$ descending to P via $\pi_{\Gamma'}$. The inclusion $\Lambda^\vee \subset (\Lambda')^\vee$ has finite index, let r be the smallest positive integer such that $rv \in \Lambda^\vee$. Since $D_v = D_{rv}$, up to multiplicities we have

$$\phi_* P \leq \pi_\Gamma \left(\bigcup_{g\in\Gamma'} D_{g(rv)} \right) \leq \sum_{\delta\in D(\Lambda)} H_{r^2 m,\delta}.$$

□

In fact, in the case of the pullback map ϕ^*, we can say something even more precise.

Proposition 4.2. *In the setting above, assume further that the Hodge class λ (resp. λ') lies in the interior of $\mathrm{Eff}^{NL}\left(\mathcal{D}/\Gamma\right)$ (resp. $\mathrm{Eff}^{NL}\left(\mathcal{D}/\Gamma\right)$), then*

$$\phi^*\partial\mathrm{Eff}^{NL}\left(\mathcal{D}/\Gamma\right) \subset \partial\mathrm{Eff}^{NL}\left(\mathcal{D}/\Gamma'\right) \quad \textit{and} \quad \phi^*\left(\mathrm{Eff}^{NL}\left(\mathcal{D}/\Gamma\right)\right)^\circ \subset \left(\mathrm{Eff}^{NL}\left(\mathcal{D}/\Gamma'\right)\right)^\circ.$$

In particular if Z is an element of $\partial\mathrm{Eff}^{NL}\left(\mathcal{D}/\Gamma\right)$, then every irreducible component of ϕ^*Z is contained in $\partial\mathrm{Eff}^{NL}\left(\mathcal{D}/\Gamma'\right)$.

Proof. Let Z be an element of the boundary $\partial\mathrm{Eff}^{NL}\left(\mathcal{D}/\Gamma\right)$. Since by assumption the Hodge class λ is contained in the interior $\left(\mathrm{Eff}^{NL}\left(\mathcal{D}/\Gamma\right)\right)^\circ$, we have that for any $\varepsilon > 0$, the class $Z - \varepsilon\lambda$ lies outside of $\mathrm{Eff}^{NL}\left(\mathcal{D}/\Gamma\right)$. If $\phi^*\left(Z - \varepsilon\lambda\right) \in \mathrm{Eff}^{NL}\left(\mathcal{D}'/\Gamma\right)$, then by Proposition 4.1 we have that the class

$$\phi_*\phi^*\left(Z - \varepsilon\lambda\right) = \deg(\phi)\left(Z - \varepsilon\lambda\right)$$

lies in $\mathrm{Eff}^{NL}\left(\mathcal{D}/\Gamma\right)$, which is a contradiction. Therefore,

$$\phi^*\left(Z - \varepsilon\lambda\right) = \phi^* Z - \varepsilon\lambda$$

lies outside of $\mathrm{Eff}^{NL}\left(\mathcal{D}/\Gamma'\right)$ for all $\varepsilon > 0$. Note that by assumption $\phi^*\lambda = \lambda'$ also lies in the interior of $\left(\mathrm{Eff}^{NL}\left(\mathcal{D}/\Gamma'\right)\right)^\circ$. Hence ϕ^*Z lies in the boundary $\partial\mathrm{Eff}^{NL}\left(\mathcal{D}/\Gamma\right)$.

Similarly, if Z lies in the interior $\left(\mathrm{Eff}^{NL}\left(\mathcal{D}/\Gamma\right)\right)^\circ$, then for $|\varepsilon| > 0$ sufficiently small, the class $Z - \varepsilon\lambda$ lies in $\mathrm{Eff}^{NL}\left(\mathcal{D}/\Gamma\right)$ and so by Proposition 4.1, we have that $\phi^*(Z - \varepsilon\lambda) = \phi^*Z - \varepsilon\phi^*\lambda$ lies in $\mathrm{Eff}^{NL}\left(\mathcal{D}'/\Gamma\right)$. It follows that ϕ^*Z is in the interior $\left(\mathrm{Eff}^{NL}\left(\mathcal{D}'/\Gamma\right)\right)^\circ$. □

Remark 4.3. When Λ splits off two copies of U and $\Gamma = \widetilde{\mathrm{O}}^+\left(\Lambda\right)$, then by [BM19, Section 4] the Hodge class is in the interior of the NL cone.

Let $\rho \in \Lambda_{\mathbb{Q}}$ and consider the reflection with respect to a negative

$$\sigma_\rho : v \mapsto v - 2\frac{\langle v, \rho\rangle}{\langle \rho, \rho\rangle}\rho \in \mathrm{O}\left(\Lambda_{\mathbb{Q}}\right). \tag{13}$$

The element ρ is called Γ-*reflexive* if $\langle \rho, \rho\rangle < 0$ and σ_ρ or $-\sigma_\rho$ is in $\Gamma \subset \mathrm{O}^+\left(\Lambda_{\mathbb{Q}}\right)$. Note that $\sigma_\rho \in \mathrm{O}^+(\Lambda_{\mathbb{Q}})$ when $\langle \rho, \rho\rangle < 0$. The same holds for $-\sigma_\rho$ since $-\mathrm{Id} \in \mathrm{O}^+(\Lambda)$ when Λ has signature $(2, n)$. Recall that the divisibility $\mathrm{div}(\rho)$ of a primitive element $\rho \in \Lambda$ is the positive generator of the ideal $\langle \rho, \Lambda\rangle \subset \mathbb{Z}$. The reflection σ_ρ is in $O(\Lambda)$ if and only if $\mathrm{div}(\rho)$ is $Q(\rho)$ or $2Q(\rho)$.

The modular projection π_Γ is simply ramified [GHS07, Theorem 2.12 and Corollary 2.13] (see also [GHS13, Section 6.2]) along the union of hyperplanes D_ρ where ρ is Γ-reflexive. Further, since $\pi_\Gamma = \phi \circ \pi_{\Gamma'}$, the finite map ϕ also simply ramified.

For $v \in \Lambda$ we denote by v_* the primitive element in $\Lambda^\vee$ given by $v/\mathrm{div}(v)$. For $\rho \in \Lambda_\mathbb{Q}$ we denote by P_ρ (resp. P'_ρ) the irreducible and reduced divisor supported on the image $\pi_\Gamma(D_\rho)$ (resp. $\pi_{\Gamma'}(D_\rho)$). In particular, $P_\rho = P_{r\rho}$, and when ρ is primitive in Λ, with $\rho_* = \rho/\mathrm{div}(\rho)$, if $(m, \mu) = (Q(\rho_*), \rho_* + \Lambda)$, then $P_\rho = P_{m,\mu}$ if $2\mu \neq 0$ in $D(\Lambda)$, and $P_\rho = \frac{1}{2}P_{m,\mu}$ otherwise. The reduced branch divisor of π_Γ is then given by

$$\mathrm{Br}\,(\pi_\Gamma) = \sum_{\substack{\mathrm{P}\Gamma-\mathrm{orb.} \\ \rho\ \Gamma-\mathrm{ref.}}} P_\rho, \tag{14}$$

where the sum runs over PΓ-orbits of reflexive primitive elements $\rho \in \Lambda$. Further, since $\pi_\Gamma = \phi \circ \pi_{\Gamma'}$, the finite map ϕ is simply ramified at

$$R_\phi = \sum_{\substack{\mathrm{P}\Gamma'-\mathrm{orb.} \\ \rho\ \Gamma-\mathrm{ref.}}} P'_\rho - \sum_{\substack{\mathrm{P}\Gamma'-\mathrm{orb.} \\ \rho\ \Gamma'-\mathrm{ref.}}} P'_\rho. \tag{15}$$

Proposition 4.4. *Let Λ be an even lattice of signature $(2, n)$. Then, the reduced branch divisor of the modular projection $\pi : \mathcal{D}_\Lambda \longrightarrow \mathcal{D}_\Lambda / \widetilde{\mathrm{O}}^+(\Lambda)$ always contains $\frac{1}{2}H_{-1,0}$.*

Proof. In light of Equation (4) the reduced Heegner divisor $\frac{1}{2}H_{-1,0}$ is the sum of $P'_{\rho_*}s$ where $\rho_* \in \Lambda^\vee$ is primitive and for some integer $r > 0$

$$r\rho_* \in \Lambda \ \text{ and } \ Q(r\rho_*) = \frac{\langle r\rho_*, r\rho_* \rangle}{2} = -1.$$

We have to show that for all such ρ_* one has $\pm\sigma_{\rho_*} \in \widetilde{\mathrm{O}}^+(\Lambda)$. Note that $\langle r\rho_*, r\rho_* \rangle = -2$ implies that $r\rho_*$ is primitive in Λ. Then the inclusion $\pm\sigma_{\rho_*} \in \widetilde{\mathrm{O}}^+(\Lambda)$ follows from [GHS07, Proposition 3.1]. □

4.1. **Pull-back formula.** The finite index embedding $\Lambda' \subset \Lambda$ corresponds to the isotropic subgroup

$$H = \Lambda / \Lambda' \subset D(\Lambda') = (\Lambda')^\vee / \Lambda'$$

with the natural projection $p : H^\perp \longrightarrow H^\perp / H = \Lambda^\vee / \Lambda$. The map

$$\begin{aligned} \psi : \mathbb{C}\,[D(\Lambda')] \quad &\longrightarrow \quad \mathbb{C}\,[D(\Lambda)] \\ e_{\mu'} \quad &\mapsto \quad \begin{cases} e_{p(\mu')} & \text{if } \mu' \in H^\perp \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

is equivariant with respect to the Weil representations ρ_Λ and $\rho_{\Lambda'}$ and induces a linear map [Ma19, Lemma 2.1 and Corollary 2.2] given on Fourier expansions by:

$$\psi : \mathrm{Mod}^\circ_{k,\Lambda'} \longrightarrow \mathrm{Mod}^\circ_{k,\Lambda}, \quad \sum_{\mu' \in D(\Lambda')} \Phi_{\mu'}(q) e_{\mu'} \mapsto \sum_{\mu \in D(\Lambda)} \left(\sum_{\substack{\mu' \in H^\perp \\ p(\mu') = \mu}} \Phi_{\mu'}(q) \right) e_\mu.$$

Then the dual map $\psi^\vee : \left(\mathrm{Mod}^\circ_{k,\Lambda}\right)^\vee \longrightarrow \left(\mathrm{Mod}^\circ_{k,\Lambda'}\right)^\vee$ acts on coefficient extraction functionals by

$$c_{m,\mu} \mapsto c_{m,\mu} \circ \psi = \sum_{\substack{\mu' \in H^\perp \\ p(\mu') = \mu}} c_{m,\mu'}.$$

In particular, via the isomorphism (6), the map on Picard groups $\mathrm{Pic}_\mathbb{Q}\left(\mathcal{D}/\Gamma\right) \longrightarrow \mathrm{Pic}_\mathbb{Q}\left(\mathcal{D}/\Gamma'\right)$ is given by $H_{m,\mu} \mapsto \sum_{\mu' \in p^{-1}(\mu)} H'_{m,\mu'}$. We observe that this map is, up to multiplicities coming from ramification, the map from geometry given by the pullback via the algebraic map $\phi : \mathcal{D}/\Gamma' \longrightarrow \mathcal{D}/\Gamma$ sending Γ'-orbits to Γ-orbits.

Proposition 4.5. *The pullback $\phi^* : \mathrm{Pic}_\mathbb{Q}\left(\mathcal{D}/\Gamma\right) \longrightarrow \mathrm{Pic}_\mathbb{Q}\left(\mathcal{D}/\Gamma'\right)$ is given on Heegner divisors up to multiplicity by*

$$\mathrm{supp}\left(\phi^* H_{m,\mu}\right) = \mathrm{supp}\left(\sum_{\mu' \in p^{-1}(\mu)} H'_{m,\mu'} \right).$$

Proof. Recall that $H_{m,\mu}$ is given by the quotient

$$H_{m,\mu} = \left(\sum_{\substack{v \in \mu + \Lambda \\ Q(v) = m}} D_v \right) / \Gamma.$$

Further the projection $p : H^\perp \longrightarrow D(\Lambda)$ induces a disjoint union decomposition of the class $\mu + \Lambda \in D(\Lambda)$ given by

$$\mu + \Lambda = \coprod_{\substack{\mu' \in H^\perp \\ p(\mu') = \mu}} \mu' + \Lambda', \quad \text{where} \quad \sum_{\substack{v \in \mu + \Lambda \\ Q(v) = m}} D_v = \sum_{\substack{\mu' \in H^\perp \\ p(\mu') = \mu}} \sum_{\substack{v \in \mu' + \Lambda' \\ Q(v) = m}} D_v.$$

Since for each $\mu' + \Lambda'$, the cycle $\sum_{\substack{v \in \mu' + \Lambda' \\ Q(v) = m}} D_v$ is Γ'-invariant and descends to $H'_{m,\mu'}$ in $\mathcal{D}/\Gamma'$. At the level of supports we have

$$\phi^{-1}\left(H_{m,\mu}\right) = \left(\bigcup_{\substack{v \in \mu + \Lambda \\ Q(v) = m}} D_v \right) / \Gamma' = \mathrm{supp}\left(\sum_{\substack{\mu' \in H^\perp \\ p(\mu') = \mu}} H_{m,\mu'} \right).$$

□

5. Moduli of K3 surfaces

As an example of how to apply the results of Sections 3 and 4, we now specialize to studying the NL-cones $\mathrm{Eff}^{NL}(\mathcal{F}_{2d})$ and the finite maps of Section 4 in the case of moduli spaces $\mathcal{F}_{2d}$ of quasi-polarized K3 surfaces. In this case $\Lambda_{2d} = U^{\oplus 2} \oplus E_8(-1)^{\oplus 2} \oplus A_1(-d)$ and if ℓ and ℓ' are the generators of $A_1(-d)$ and $A_1(-dr^2)$ respectively, the finite index embedding $\Lambda_{2dr^2} \hookrightarrow \Lambda_{2d}$ is given by $\ell' \mapsto r\ell$. Further $\Gamma = \widetilde{\mathrm{O}}^+(\Lambda_{2d})$ and $\Gamma' = \widetilde{\mathrm{O}}^+(\Lambda_{2dr^2})$ (see [GHS13, Lemma 7.1]), and the map ϕ in (12) is on coarse moduli spaces $\phi : \mathcal{F}_{2dr^2} \longrightarrow \mathcal{F}_{2d}$. We will denote by $\pi_r : \mathcal{D} \longrightarrow \mathcal{F}_{2dr^2}$ and $\pi : \mathcal{D} \longrightarrow \mathcal{F}_{2d}$ the corresponding modular projections and $\{e, f\}$ will denote the standard basis of the first copy of U. From [GHS07, Corollary 3.4], one has that a primitive element $\rho \in \Lambda_{2d}$ with $\langle \rho, \rho \rangle < 0$ is Γ-reflexive if and only if

$$\langle \rho, \rho \rangle = -2 \quad \text{or} \quad \langle \rho, \rho \rangle = -2d \ \text{ and } \ \mathrm{div}(\rho) \in \{d, 2d\}. \tag{16}$$

For $\rho = \lambda x + m\ell$ with $x \in U^{\oplus 2} \oplus E_8(-1)^{\oplus 2}$ primitive, the divisibility is $\mathrm{div}(\rho) = \gcd(\lambda, 2dm)$ and $\rho_* = m \cdot \frac{2d}{\mathrm{div}(\rho)} \ell_*$.

Corollary 5.1. *The reduced branch divisor of the modular projection $\pi : \mathcal{D} \longrightarrow \mathcal{F}_{2d}$ in terms of primitive Heegner divisors $P_{\Delta,\delta}$ is given by*

$$\begin{aligned} \mathrm{Br}(\pi) = {} & \frac{1}{2}\left(P_{-1,0} + N_d \cdot P_{-\frac{1}{4}, d\ell_*} + M_d \cdot P_{-\frac{1}{2}, 2\ell_*}\right) \\ & + \sum_{\substack{0 \le m < d/2 \\ m^2 \equiv 1 \mod d}} P_{-\frac{1}{d}, 2m\ell_*} + \sum_{\substack{0 \le m < d \\ m^2 \equiv 1 \mod 4d}} P_{-\frac{1}{4d}, m\ell_*}, \end{aligned} \tag{17}$$

with $M_d = 1$ if $d = 2$ and zero otherwise, and

$$N_d = \begin{cases} 1 & d \equiv 1 \mod 4 \\ 0 & \textit{otherwise.} \end{cases}$$

Further, the ramification divisor of $\phi : \mathcal{F}_{2dr^2} \longrightarrow \mathcal{F}_{2d}$ in terms of primitive Heegner divisors P' is given by

$$R_\phi = \sum_{\substack{\mathrm{P}\Gamma'-orb. \\ \langle \frac{\rho}{s}, \frac{\rho}{s} \rangle = -2}} P'_\rho + \sum_{\substack{\mathrm{P}\Gamma'-orb. \\ \langle \frac{\rho}{s}, \frac{\rho}{s} \rangle = -2d \\ \mathrm{div}\left(\frac{\rho}{s}\right) = d}} P'_\rho + \sum_{\substack{\mathrm{P}\Gamma'-orb. \\ \langle \frac{\rho}{s}, \frac{\rho}{s} \rangle = -2d \\ \mathrm{div}\left(\frac{\rho}{s}\right) = 2d}} P'_\rho - \mathrm{Br}(\pi_r)$$

where the sums run over $\mathrm{P}\Gamma'$-orbits of primitive elements $\rho \in \Lambda_{2dr^2}$,

$$s = s(\rho) = \gcd\left(r, \mathrm{div}_{\Lambda_{2dr^2}}(\rho)\right)$$

and $\mathrm{Br}(\pi_r)$ is given by the same expression as (17) replacing d with dr^2, P with P', and ℓ with ℓ'.

Remark 5.2. Recall that for $\rho \in \Lambda_{\mathbb{Q}}$ we denote by P_ρ (resp. P'_ρ) the irreducible and reduced divisor supported on the image $\pi_\Gamma(D_\rho)$ (resp. $\pi_{\Gamma'}(D_\rho)$) and when ρ is primitive in Λ, with $\rho_* = \rho/\mathrm{div}(\rho)$, if $(m,\mu) = (Q(\rho_*), \rho_* + \Lambda)$, then $P_\rho = P_{m,\mu}$ if $2\mu \neq 0$ in $D(\Lambda)$, and $P_\rho = \frac{1}{2}P_{m,\mu}$ otherwise.

Proof of Corollary 5.1. This is a direct consequence of (14) and (16). Since $D(\Lambda_{2d})$ is cyclic generated by the class of $\ell_* = \frac{\ell}{2d}$, when $\langle\rho,\rho\rangle = -2$ the orbit of ρ only depends on its divisibility, which is either 1 or 2, the second case occurring only when $d \equiv 1 \mod 4$. Two orbit representatives are then $e - f$ and $2\left(e + \frac{d-1}{4}f\right) + \ell$, with corresponding indexing pairs $(Q(\rho_*), \rho_* + \Lambda_{2d})$ given by $(-1, 0)$ and $\left(-\frac{1}{4}, d\ell_*\right)$. If instead $\langle\rho,\rho\rangle = -2d$, imposing $\mathrm{div}(\rho) = d$ forces $\rho = \lambda dx + m\ell$ with $m^2 \equiv 1 \pmod d$ and $\rho_* = 2m\ell_*$. Conversely, for each such $0 \leq m < d$ an orbit representative is $d\left(e + \frac{m^2-1}{d}f\right) + m\ell$. The argument is similar for $\mathrm{div}(\rho) = 2d$.

We obtain in this way all the terms in (14). to take multiplicity into account, observe that $P_\rho = P_{-\rho}$ and that $P_{m,\mu}$ has multiplicity two when $2\delta_* = 0$, in particular, $P_{-\frac{1}{d},2\ell_*}$ has multiplicity 2 if and only if $d = 2$.

For the second equation, note that $\frac{\rho}{s} = \lambda\left(\frac{\mathrm{div}_{\Lambda_{2\mathrm{d}r^2}}(\rho)}{s}\right)x + m\frac{r}{s}\ell$ is primitive in Λ_{2d} and Γ-reflexive. The equation then follows from (15). □

5.1. **Computation of $\mathrm{Eff}^{NL}(\mathcal{F}_{2d})$ in low degree.** The method of Section 3 together with the bounds of Theorem 3.6 (and their Sage implementation [Wila]) allow us to compute the generating rays of $\mathrm{Eff}^{NL}(\mathcal{F}_{2d})$ for low d. These calculations confirm (aside from one additional generator in the case $d = 13$) the predictions of [Pet15, Remark 4.7.1 and Table 4.5]. We record these calculations in Table 1.

We remark that, using the formula $K_{\mathcal{F}_{2d}} = 19\lambda - \frac{1}{2}\mathrm{Br}(\pi)$ together with the first part of Corollary 5.1, one can express the canonical divisor $K_{\mathcal{F}_{2d}}$ in terms of generating rays of $\mathrm{Eff}^{NL}(\mathcal{F}_{2d})$, thereby determining the positioning of $K_{\mathcal{F}_{2d}}$ with respect to $\mathrm{Eff}^{NL}(\mathcal{F}_{2d})$. These calculations already appear in Peterson [Pet15, Table 4.5]. For all values of d appearing in Table 1 (namely for $1 \leq d \leq 20$) the canonical divisor $K_{\mathcal{F}_{2d}}$ lies outside of $\mathrm{Eff}^{NL}(\mathcal{F}_{2d})$.

5.2. **Boundary divisors in degree $2dr^2$.** In the case of the moduli space of quasi-polarized K3 surfaces $\mathcal{F}_{2d}$, we consider for any positive integer $r > 0$ the finite map $\phi\colon \mathcal{F}_{2dr^2} \to \mathcal{F}_{2d}$ induced, as in Section 4 by the embedding of lattices $\Lambda_{2dr^2} \hookrightarrow \Lambda_{2d}$ is given by $\ell' \mapsto r\ell$. It follows from Proposition 4.2 that the pullback along the finite map ϕ of any divisor lying in the boundary of $\mathrm{Eff}^{NL}(\mathcal{F}_{2d})$ lies in the boundary of $\mathrm{Eff}^{NL}(\mathcal{F}_{2dr^2})$. In particular, the pullback of any generating ray appearing in Table 1 lies in the boundary of $\mathrm{Eff}^{NL}(\mathcal{F}_{2dr^2})$.

Proposition 5.3. *Let $P_{m,\mu}$ be a primitive Heegner divisor on $\mathcal{F}_{2d}$. Let γ be the order of μ in $D(\Lambda_{2d})$ and write $\mu = \frac{2d}{\gamma}a\ell_*$, where $\gcd(a,\gamma) = 1$. Then, letting $r > 0$ be a positive integer, the support of the pullback $\phi^*P_{m,\mu}$ under the finite map $\phi\colon \mathcal{F}_{2dr^2} \to \mathcal{F}_{2d}$ is given* *by the formula*

$$\operatorname{supp}(\phi^* P_{m,\mu}) = \bigcup_{(k,s)\in I(m,\mu)} P_{\frac{m}{k^2}, \frac{2dr(a+\gamma s)}{\gamma k}\ell'_*},$$

where the index set $I(m,\mu) \subset \mathbb{Z}\times\mathbb{Z}$ is given by all $(k,s)\in\mathbb{Z}\times\mathbb{Z}$ such that

(i) $1 \le k \le r$ *and k divides* $\frac{r}{\gcd(r,\frac{2d}{\gamma})}$,

(ii) $0 \le s < kr$, *subject to the condition that* $\gcd\left(k, \frac{2d}{\gamma}(a+\gamma s)\right) = 1$ *and*

$$\frac{d}{\gamma}(a+\gamma s)^2 \equiv -\gamma m \mod \gamma k^2,$$

where the last congruence denotes that the difference of the two rational numbers is an integer multiple of γk^2.

Proof. We may choose a primitive representative $\rho \in \Lambda_{2d}$ of the $\widetilde{\mathrm{O}}^+(\Lambda_{2d})$-orbit of elements in $P_{m,\mu}$ of the form $\rho = \gamma(e+tf) + a\ell$, where ρ has divisibility γ and $t = m - \frac{d}{\gamma^2}a^2$, so that $\langle\rho,\rho\rangle = 2\gamma^2 m$ (and thus $Q(\rho_*) = \frac{\langle\rho,\rho\rangle}{2\gamma^2} = m$). By primitivity, $\gcd(\gamma, a) = 1$.

Now a primitive element $v = \gamma k x + \alpha\ell \in \Lambda_{2d}$ with $k \in \mathbb{Z}_{>0}$, $x \in U^{\oplus 2}\oplus E_8(-1)^{\oplus 2}$ primitive, and $\operatorname{div}(v) = \gamma$ is in the same $\widetilde{\mathrm{O}}^+(\Lambda_{2d})$-orbit as ρ if and only if $v_* = \rho_*$ and $\langle v,v\rangle = \langle\rho,\rho\rangle$. Namely v is in the same $\widetilde{\mathrm{O}}^+(\Lambda_{2d})$-orbit as ρ if and only if $v_* = \frac{2d}{\gamma}\alpha\ell_*$ is congruent to $\frac{2d}{\gamma}a\ell_*$ modulo Λ_{2d} and $2\gamma^2k^2\frac{\langle x,x\rangle}{2} - 2d\alpha^2 = 2\gamma^2 m$. The condition $\frac{2d}{\gamma}\alpha \equiv \frac{2d}{\gamma}a$ mod $2d$ is satisfied if and only if $\alpha = (a+\gamma s)$ for some integer s.

Therefore such a v exists if and only if one can produce a primitive $x \in U^{\oplus 2}\oplus E_8(-1)^{\oplus 2}$ such that

$$2\gamma^2k^2\frac{(x,x)}{2} - 2d(a+\gamma s)^2 = 2\gamma^2 m.$$

This is equivalent to the condition

$$\frac{d}{\gamma}(a+\gamma s)^2 \equiv -\gamma m \mod \gamma k^2. \tag{18}$$

Moreover, note that the condition $\operatorname{div}(v) = \gamma$ implies

$$\gcd\left(k, \frac{2d}{\gamma}(a+\gamma s)\right) = 1.$$

Further, the primitivity of ρ ensures $\gcd(\gamma k, a+\gamma s) = 1$ and the primitivity of v. In particular, elements of Λ_{2d} in the same $\widetilde{\mathrm{O}}^+(\Lambda_{2d})$-orbit as ρ are all of the form

$$v_{k,s} = \gamma k x + (a+\gamma s)\ell, \tag{19}$$

for any choice of integers k, s and $x \in U^{\oplus 2}\oplus E_8(-1)^{\oplus 2}$ primitive, subject to the condition

$$\gcd\left(k, \frac{2d}{\gamma}(a+\gamma s)\right) = 1 \text{ and } \frac{d}{\gamma}(a+\gamma s)^2 \equiv -\gamma m \mod \gamma k^2. \tag{20}$$

For such a choice of k, s, let $\lambda = \gcd(r, a+s\gamma)$ and consider the element $w_{k,s} = \frac{r}{\lambda} v_{k,s}$, see (19). Observe that

$$w_{k,s} = \frac{r}{\lambda}\gamma k x + \frac{a+s\gamma}{\lambda}\ell'$$

is primitive in Λ_{2dr^2}, since $\gcd(\frac{r}{\lambda}\gamma k, \frac{a+s\gamma}{\lambda}) = 1$ (using that $\gcd(k, \frac{2d}{\gamma}(a+\gamma s)) = 1$ and $\gcd(\frac{a+s\gamma}{\lambda}, \frac{r}{\lambda}) = 1$). Moreover, note that $\lambda w_{k,s} \in r \cdot \Lambda_{2d} \subset \Lambda_{2dr^2}$. Then we have

$$\langle w_{k,s}, w_{k,s}\rangle = \frac{r^2}{\lambda^2}\langle v_{k,s}, v_{k,s}\rangle = \frac{r^2}{\lambda^2}\langle v, v\rangle = \frac{2r^2\gamma^2 m}{\lambda^2}.$$

Further, the divisibility of $w_{k,s}$ is given by

$$\gcd\left(\frac{r}{\lambda}\gamma k, 2dr^2\frac{a+s\gamma}{\lambda}\right) = \frac{r\gamma}{\lambda}\gcd\left(k, \frac{2dr}{\gamma}(a+s\gamma)\right) = \frac{r\gamma}{\lambda}\gcd(k,r).$$

It follows that

$$(w_{k,s})_* = \frac{1}{\frac{r\gamma}{\lambda}\gcd(k,r)} \cdot \frac{a+s\gamma}{\lambda}\ell' = \frac{a+s\gamma}{r\gamma\gcd(k,r)}\ell' = \frac{a+s\gamma}{\gamma\gcd(k,r)}2dr\ell'_*$$
$$Q\left((w_{k,s})_*\right) = \frac{2r^2\gamma^2 m}{\lambda^2} \cdot \frac{1}{2(\frac{r\gamma}{\lambda}\gcd(k,r))^2} = \frac{m}{\gcd(k,r)^2}.$$

Therefore, the choice of integers s, k corresponds to the component of $\phi^* P_{m,\mu}$ given by

$$P_{\frac{m}{\gcd(k,r)^2}, \frac{2dr(a+s\gamma)}{\gamma\gcd(k,r)}\ell'_*}.$$

However, if the pair (s,k) satisfies (20), then so does the pair $(s, \gcd(k,r))$. Since both (s,k) and $(s,\gcd(k,r))$ correspond to the same component $P_{\frac{m}{\gcd(k,r)^2}, \frac{2dr(a+s\gamma)}{\gamma\gcd(k,r)}\ell'_*}$ of $\phi^* P_{m,\mu}$, we may restrict ourselves to pairs (s,k) as above where k divides r. Further, since $\gcd\left(k, \frac{2d}{\gamma}(a+\gamma s)\right) = 1$, the integer k is coprime with $\frac{2d}{\gamma}$ and we may restrict further to the case that k divides $\frac{r}{\gcd\left(r, \frac{2d}{\gamma}\right)}$. Such a pair (s,k) then yields uniquely to the component of $\phi^* P_{m,\mu}$ given by

$$P_{\frac{m}{k^2}, \frac{2dr(a+s\gamma)}{\gamma k}\ell'_*}.$$

Similarly, we can reach all possible elements $\frac{2dr(a+s\gamma)}{\gamma k}\ell'_*$ by taking $0 \le s < kr$. Note that the condition (20) on (s,k) depends only on the value of s modulo k^2. Since we have assumed that k divides r, we thus may restrict to the case $0 \le s < kr$. This gives us the formula for the support of $\phi^* P_{m,\mu}$. □

Example 5.4. We consider the case of $\mathcal{F}_2$ and apply Proposition 5.3 to compute the pullback $\phi^* P_{-1,0}$ under the finite map $\phi\colon F_{2dr^2} \to \mathcal{F}_2$ for some low values of r in Table 2.

Proof of Theorem 1.3. The theorem follows immediately from Proposition 5.3 together with Proposition 4.2 and the computations of Table 1. □

6. Hyperkähler fourfolds of $\mathrm{K3}^{[2]}$-type

Let (X, L) be a primitively polarized hyperkähler fourfold of $\mathrm{K3}^{[2]}$-type. Then the Beauville–Bogomolov–Fujiki lattice $(H^2(X,\mathbb{Z}), q_X)$ is isomorphic to

$$\Lambda = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus A_1(-1).$$

The polarization L comes with two invariants singling out a component of the moduli space. These are the Beauville–Bogomolov–Fujiki degree $2d$ and the divisibility $\gamma \in \{1,2\}$. Further, when $\gamma = 2$, then $d = 4t-1$ for some $t \geq 1$. We will denote by $\mathcal{M}^{\gamma}_{\mathrm{K3}^{[2]},2d}$ the partial compactification of the corresponding moduli space given by the modular variety $\mathcal{D}_{\Lambda_h}/\mathrm{Mon}^2(\Lambda, h)$, where after choosing a marking, Λ_h is the orthogonal complement of $h = c_1(L)$ in Λ, and $\mathrm{Mon}^2(\Lambda, h) = \widetilde{\mathrm{O}}^+(\Lambda_h)$, cf. [Mar11, Lemma 9.2] and [BBBF23, Proposition 3.7].

Unirational parametrizations of these moduli spaces are only available in degree $2d = 2$ for $\gamma = 1$, and $t = 1,3,5$ with $2d = 8t-2$ for $\gamma = 2$, cf. [BD85, O'G06, IR01, IR07, DV10], see also [Mon13, Proposition 1.4.1]. Further, from [GHS10, BBBF23] it follows that

$$\mathrm{Kod}\left(\mathcal{M}^{1}_{\mathrm{K3}^{[2]},2d}\right) = \begin{cases} -\infty & \text{if } d = 1 \\ 20 & \text{if } d \geq 12, \end{cases} \quad \mathrm{Kod}\left(\mathcal{M}^{2}_{\mathrm{K3}^{[2]},8t-2}\right) = \begin{cases} -\infty & \text{if } t = 1,3,5 \\ 20 & \text{if } t \geq 12, t = 10. \end{cases}$$

With this in mind, in addition to computing the NL cones of these moduli spaces, we aim to understand the positivity properties of their canonical classes. Recall that the canonical class for these moduli spaces $\mathcal{M}^{\gamma}_{\mathrm{K3}^{[2]},2d}$ is given by the formula

$$K_{\mathcal{M}^{\gamma}_{\mathrm{K3}^{[2]},2d}} = 20\lambda - \frac{1}{2}\mathrm{Br}\left(\pi_d\right), \tag{21}$$

where λ is the Hodge class and $\mathrm{Br}(\pi_d)$ is the branch divisor of the projection $\mathcal{D} \longrightarrow \mathcal{D}_{\Lambda_h}/\widetilde{\mathrm{O}}^+(\Lambda_h)$.

We will denote by Λ_d and Λ_t the lattices $U^{\oplus 2} \oplus E_8(-1)^{\oplus 2} \oplus Q_d$ (resp. Q_t) where

$$Q_d = \mathbb{Z}\ell + \mathbb{Z}\delta = \begin{pmatrix} -2d & 0 \\ 0 & -2 \end{pmatrix} \quad \text{and} \quad Q_t = \mathbb{Z}u + \mathbb{Z}v = \begin{pmatrix} -2t & 1 \\ 1 & -2 \end{pmatrix}.$$

These correspond to the lattice Λ_h for (X, L) in $\mathcal{M}^{\gamma}_{\mathrm{K3}^{[2]},2d}$ when $\gamma = 1$, respectively $\gamma = 2$ with $d = 4t-1$. When $\gamma = 1$, the discriminant group is isomorphic to $\mathbb{Z}/2d\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, generated by ℓ_* and δ_*. When $\gamma = 2$, the discriminant group is isomorphic to $\mathbb{Z}/d\mathbb{Z}$ and is generated by $(2u+v)_*$.

The computation of $\mathrm{Br}(\pi_d)$ requires the following lemma classifying reflexive elements.

Lemma 6.1 (Proposition 3.2 and Corollary 3.3 in [GHS07])**.** *A primitive element $\rho \in \Lambda_d$ with $\langle \rho, \rho \rangle < 0$ is reflexive if and only if one of the following holds*

(i) $\langle \rho, \rho \rangle = -2$,

(ii) $\langle \rho, \rho \rangle = -2d$ *and* $\mathrm{div}(\rho) = 2d$,

(iii) *d is odd, $\langle\rho,\rho\rangle = -2d$ and $\operatorname{div}(\rho) = d$,*

(vi) *d is even, $\langle\rho,\rho\rangle = -2d$, $\operatorname{div}(\rho) = d$, and $\rho_* \equiv 2m\ell_* \mod \Lambda_d$ with $m^2 \equiv 1 \pmod d$.*

Further, a primitive element $\rho \in \Lambda_t$ with $\langle\rho,\rho\rangle < 0$ is reflexive if and only if one of the following holds

(i) *$\langle\rho,\rho\rangle = -2$ and $\operatorname{div}(\rho) = 1$,*

(ii) *$\langle\rho,\rho\rangle = -2d$ and $\operatorname{div}(\rho) = d$,*

where $d = 4t - 1$.

Proof. The only case which is not treated in [GHS07] is when $\gamma = 1$, $d > 1$ is even, $\langle\rho,\rho\rangle = -2d$, and $\operatorname{div}(\rho) = d$. In this case (see [GHS07, Equation (18)] the vector ρ being reflexive is equivalent to

$$2v \equiv -\langle\rho,r\rangle\rho_* \mod \Lambda_d \text{ for } v \in \{\ell_*, \delta_*\}. \tag{22}$$

Since the divisibility of ρ is d, we can assume $\rho = dx+m\ell+\frac{d}{2}\lambda\delta$, with $x \in U^{\oplus 2}\oplus E_8(-1)^{\oplus 2}$ and $m,\lambda \in \mathbb{Z}$. For $v = \delta_*$, Condition (22) becomes $0 \equiv \langle\rho,\delta_*\rangle \equiv \frac{1}{2}\langle\rho,\delta\rangle \equiv -\frac{d}{2}\lambda\rho_*$. Since ρ is primitive, the class ρ_* has order d, hence λ has to be even and $\rho_* \equiv 2m\ell_*+\lambda\delta_* \equiv 2m\ell_* \mod \Lambda_d$. With $v = \delta_*$, Condition (22) implies $2\ell_* \equiv -2m\langle\rho,\ell_*\rangle\ell_* \equiv -\frac{m}{d}\langle\rho,\ell\rangle\ell_* \equiv 2m^2\ell_*$. Since $\langle\ell_*\rangle \cong \mathbb{Z}/2d\mathbb{Z}$, the condition holds if and only if $m^2 \equiv 1 \pmod d$. □

The equation (14) gives us the following classes in terms of primitive Heegner divisors.

Proposition 6.2. *Let π_d (resp. π_t) be the modular projection $\mathcal{D}_{\Lambda_d} \longrightarrow \mathcal{D}_{\Lambda_d}/\widetilde{\mathrm{O}}^+(\Lambda_d)$ (resp. for Λ_t). Then, the reduced branch divisor is given by*

$$\begin{aligned}\operatorname{Br}(\pi_d) =& \frac{1}{2}\left(P_{-1,0} + P_{-\frac{1}{4},\delta_*} + C_{1,d}\cdot P_{-\frac{1}{4},d\ell_*+\delta_*} + C_{2,d}\cdot P_{-\frac{1}{4},d\ell_*} + M_d\cdot P_{-\frac{1}{2},2\ell_*}\right)\\ &+ \sum_{\substack{0\le m<d/2\\ m^2\equiv 1 \mod d}} P_{-\frac{1}{d},2m\ell_*} + \sum_{\substack{0\le m<d\\ m^2\equiv 1 \mod 4d}} P_{-\frac{1}{4d},m\ell_*}\\ &+ C_{2,d}\cdot \sum_{\substack{0\le m<d/2\\ 4m^2\equiv 1 \mod d\\ \frac{4m^2-1}{d}\equiv 3 \mod 4}} P_{\frac{-1}{4d},2m\ell_*+\delta_*} + \sum_{\substack{0\le m<d\\ m^2\equiv 1 \mod d\\ \frac{m^2-1}{d}\equiv 3 \mod 4}} P_{-\frac{1}{4d},m\ell_*+\delta_*}.\end{aligned}$$

Where $M_d = 1$ if $d = 2$ and zero otherwise, and

$$C_{1,d} = \begin{cases}1 & d\equiv 0 \mod 4\\ 0 & \textit{otherwise}\end{cases} \quad \textit{and} \quad C_{2,d} = \begin{cases}1 & d\equiv 1 \mod 4\\ 0 & \textit{otherwise.}\end{cases}$$

Similarly, for π_t one has:

$$\operatorname{Br}(\pi_t) = \frac{1}{2}P_{-1,0} + \sum_{\substack{0<m<d/2\\ m^2\equiv 1 \mod d}} P_{-\frac{1}{d},m(2u+v)_*},$$

where $d = 4t - 1$.

Remark 6.3. Note that the last summand of $\mathrm{Br}\,(\pi_d)$ is zero when $d \not\equiv 0 \mod 8$.

Proof of Proposition 6.2. The proof follows from Equation (14) and Lemma 6.1. As in Corollary 5.1, it is an elementary case-by-case analysis. By Eichler's Criterion, for each square and divisibility $(2r, \gamma)$ singled out in Lemma 6.1 there is a contribution to the sum for each orbit of a primitive element with the given square and divisibility or, equivalently, for each pair $(\langle\rho,\rho\rangle, \rho_*) \in 2\mathbb{Z} \times D(\Lambda)$ with $\langle\rho,\rho\rangle = 2r$ and $\mathrm{ord}(\rho_*) = \gamma$. For the sake of brevity, we will only treat the first case, the rest are analogous. If $\langle\rho,\rho\rangle = -2$, then $\mathrm{div}(\rho) \in \{1, 2\}$. If $\mathrm{div}(\rho) = 1$, then $\rho_* = 0$ in $D(\Lambda_d)$ and there is only one orbit. One can take as representative $\rho = e - f$. This accounts for $P_{-1,0}$. If $\mathrm{div}(\rho) = 2$, then ρ_* has order two in $D(\Lambda_d)$:

$$\rho_* \in \{\delta_*, d\ell_*, \delta_* + d\ell_*\}.$$

If $\rho_* = \delta_*$, then one can take as orbit representative $\rho = \delta$. This accounts for the contribution $P_{\frac{1}{4},\delta_*}$. If $\rho_* = d\ell_*$, then ρ is of the form $2x+\alpha\delta+\beta\ell$ with $x \in U^{\oplus 2} \oplus E_8(-1)^{\oplus 2}$, α even and β odd. Since $\langle\rho,\rho\rangle = -2$, this forces $d \equiv 1 \mod 4$. Then one can take as orbit representative $\rho = 2\left(e + \frac{d-1}{4} f\right) + \ell$. This accounts for the contribution of $C_{2,d} \cdot P_{-\frac{1}{4},d\ell_*}$. Similarly, if $\rho_* = \delta_* + d\ell_*$, then ρ is of the form $2x + \alpha\delta + \beta\ell$ with both α, β odd. In this case $\langle\rho,\rho\rangle = -2$ forces $d \equiv 0 \mod 4$, and one can take as representative $\rho = 2\left(e + \frac{d}{4} f\right) + \delta + \ell$, accounting for the contribution $C_{1,d} \cdot P_{-\frac{1}{4},d\ell_*+\delta_*}$. The other cases are treated similarly. □

Example 6.4. As an example of these computations of the branch divisor of $\mathrm{Br}\,(\pi_d)$, in Table 3 we list the class of the branch divisor of π_d in the split case for low polarization degrees in terms of primitive Heegner divisors as well as a linear combination of Heegner divisors generating the Picard group.

Using the description of $\mathrm{Br}\,(\pi_d)$ in Proposition 6.2, one can compute the canonical classes of the moduli spaces $\mathcal{M}^{\gamma}_{\mathrm{K3}^{[2]},2d}$.

Theorem 6.5. *The generators of the NL-cone* $\mathrm{Eff}^{NL}\left(\mathcal{M}^{\gamma}_{\mathrm{K3}^{[2]},2d}\right)$ *as well as the position of the canonical class for* $d \leq 5$ *in the split case, and* $t \leq 5$ *with* $d = 4t - 1$ *in the non-split case are as appear in Tables 4 and 5.*

Proof. The description of the generators of the NL-cones $\mathrm{Eff}^{NL}\left(\mathcal{M}^{\gamma}_{\mathrm{K3}^{[2]},2d}\right)$ is computed using the Sage program [Wila] following the procedure described in Section 3. The positivity of $K_{\mathcal{M}^{\gamma}_{\mathrm{K3}^{[2]},2d}}$ is calculated using Proposition 6.2 and (21) together with the explicit descriptions of the NL-cones.

□

7. Moduli of Kum_n and OG6-type

Let (X, L) be a primitively polarized hyperkähler sixfold where X is deformation equivalent to O'Grady's six-dimensional example [O'G03]. In this case the Beauville–Bogomolov–Fujiki lattice $(H^2(X,\mathbb{Z}), q_X)$ is isomorphic [Rap08] to $\Lambda = U^{\oplus 3} \oplus A_1(-1)^{\oplus 2}$. Further, the monodromy group coincides [MR21] with the full group $\mathrm{O}^+(\Lambda)$. If $h = c_1(L) \in \Lambda$ with $(h,h) = 2d > 0$, then $\gamma = \mathrm{div}_\Lambda(h)$ can be 1 or 2.

We denote by Λ_h be the orthogonal complement of h in Λ. The period domain $\mathcal{M}^{\gamma}_{\mathrm{OG6},2d} = \mathcal{D}_{\Lambda_h}/\mathrm{O}^+(\Lambda, h)$ is a partial compactification of the moduli space parameterizing primitively polarized hyperkähler sixfolds of OG6-type with a polarization of degree $2d$ and divisibility γ. It is always irreducible [Son23, Section 3] and when $\gamma = 1$, is non-empty for all $d \geq 1$ and when $\gamma = 2$, is non-empty only for $d \equiv 2, 3 \mod 4$. Not much is known about the global geometry of the moduli spaces $\mathcal{M}^{\gamma}_{\mathrm{OG6},2d}$.

In the *split case* $\gamma = 1$, $\Lambda_h \cong U^{\oplus 2} \oplus A_1(-1)^{\oplus 2} \oplus A_1(-d)$. When $\gamma = 2$, then $\Lambda_h = U^{\oplus 2} \oplus Q_t$, where

$$Q_t = \begin{cases} A_1(-1) \oplus \begin{pmatrix} -2 & 1 \\ 1 & -2t \end{pmatrix} & \text{when } d = 4t-1 \\ \begin{pmatrix} -2 & 0 & 1 \\ 0 & -2 & 1 \\ 1 & 1 & -2t \end{pmatrix} & \text{when } d = 4t-2. \end{cases}$$

We denote by δ_1, δ_2 the generators of the two copies of $A_1(-1)$ in Λ, by $\{e, f\}$ and $\{e_1, f_1\}$ the canonical basis of two orthogonal copies of the hyperbolic plane.

Lemma 7.1. *The polarized monodromy group* $\mathrm{Mon}^2(\Lambda, h) \subset \mathrm{O}^+(\Lambda_h)$ *is a double extension of the stable orthogonal group of* Λ_h. *More precisely*

$$\mathrm{O}^+(\Lambda, h) = \langle \widetilde{\mathrm{O}}^+(\Lambda_h), \sigma_\kappa \rangle,$$

where

$$\kappa = \begin{cases} \delta_1 - \delta_2 & \text{if } \gamma = 1, \\ f + (e_1 + f_1) + \delta_1 - \delta_2 & \text{if } \gamma = 2, d = 4t-1, \\ \delta_1 - \delta_2 & \text{if } \gamma = 2, d = 4t-2. \end{cases}$$

Proof. Let $h \in \Lambda$ be an element with $\langle h, h \rangle = 2d$ and $\mathrm{div}_\Lambda(h) = \gamma$. Since Λ and Λ_h contain two copies of the hyperbolic plane, the map $\mathrm{O}(\Lambda) \longrightarrow \mathrm{O}(D(\Lambda))$ and the respective one for Λ_h are surjetive. In particular, $\left[\mathrm{O}^+(\Lambda) : \widetilde{\mathrm{O}}^+(\Lambda)\right] = 2$. Since $\widetilde{\mathrm{O}}^+(\Lambda, h) = \widetilde{\mathrm{O}}^+(\Lambda_h)$, see for example [ABL24, Lemma 3.13], either $\mathrm{O}^+(\Lambda_h)$ is equal to $\widetilde{\mathrm{O}}^+(\Lambda_h)$ or it is a double extension.

By Eichler's Criterion we can always assume $h = e + df$ when $\gamma = 1$, and $h = 2(e+tf) - \delta_1$ or $h = 2(e+tf) - (\delta_1 + \delta_2)$ when $\gamma = 2$. Assume $\gamma = 1$. Since $\delta_1 - \delta_2 \in \Lambda_h$, the reflection $\sigma_{\delta_1-\delta_2} \in \mathrm{O}^+(\Lambda)$ fixes h and exchanges the two generators of $D(\Lambda)$, so $\sigma_{\delta_1-\delta_2} \in \mathrm{O}^+(\Lambda, h)$ and $\sigma_{\delta_1-\delta_2} \notin \widetilde{\mathrm{O}}^+(\Lambda_h)$. The others are analogous. □

Theorem 7.2. *The moduli space $\mathcal{M}^{\gamma}_{\mathrm{OG6},2d}$ is uniruled in the following cases*

(i) *when $\gamma = 1$ for $d \leq 12$,*
(ii) *when $\gamma = 2$ for $t \leq 10$ and $t = 12$ with $d = 4t - 1$,*
(iii) *when $\gamma = 2$ for $t \leq 9$ and $t = 11, 13$ with $d = 4t - 2$.*

Proof. We will show that $\mathcal{M}_h = \mathcal{D}_{\Lambda_h} / \widetilde{\mathrm{O}}^+(\Lambda_h)$ is uniruled. Since $\widetilde{\mathrm{O}}^+(\Lambda_d) \subset \mathrm{Mon}^2(\Lambda, h)$, there is a dominant map $\mathcal{M}_h \longrightarrow \mathcal{M}^{\gamma}_{\mathrm{OG6},2d}$ giving us uniruledness for the moduli space. Let $\overline{\mathcal{M}}_h^{\mathrm{tor}}$ be a toroidal compactification of $\mathcal{M}_h$. Toroidal compactifications of locally symmetric manifolds of type $\mathrm{O}(2,n)$ are normal and have at worst finite-quotient-singularities, in particular, rational and $\mathbb{Q}$-factorial, see [AMRT10]. Riemann-Hurwitz shows that $K_{\overline{\mathcal{M}}_h^{\mathrm{tor}}} = 5\lambda - \frac{1}{2}\mathrm{Br}(\pi) - b\delta$, where δ is the boundary divisor and the value of $b > 0$ depends on the choice of the toroidal compactification and the ramification at the boundary. Since λ is ample on the Baily-Borel compactification $\epsilon : \overline{\mathcal{M}}_h^{\mathrm{tor}} \longrightarrow \overline{\mathcal{M}}_h^{\mathrm{BB}}$, and $\overline{\mathcal{M}}_h^{\mathrm{BB}} \setminus \mathcal{M}_h$ is one dimensional, we can choose a representative for the class $(\epsilon^*\lambda)^4$ which does not meet the boundary of $\overline{\mathcal{M}}_h^{\mathrm{tor}}$. The curve class $(\epsilon^*\lambda)^4$ is nef and intersects trivially δ. Let $\eta : Y_h \longrightarrow \overline{\mathcal{M}}_h^{\mathrm{tor}}$ be a smoothing. Note that $\eta^*(\epsilon^*\lambda)^4$ is nef and if $K_{Y_h} \cdot \eta^*(\epsilon^*\lambda)^4 < 0$, then K_{Y_h} is not pseudo-effective. By [BDPP13] this would imply that Y_h (and therefore $\mathcal{M}_h$) is birationally covered by rational curves. Note that the projection formula and the fact that $\delta \cdot (\epsilon^*\lambda)^4 = 0$ implies

$$K_{Y_h} \cdot \eta^* (\epsilon^*\lambda)^4 = \left(5\lambda - \frac{1}{2}\mathrm{Br}(\pi)\right) \cdot (\epsilon^*\lambda)^4 .$$

Since the curve $(\epsilon^*\lambda)^4$ does not intersect the boundary, if $D \subset \mathcal{M}_h$ is a divisor and $\overline{D}$ the closure of D in $\overline{\mathcal{M}}_h^{\mathrm{tor}}$, then, the intersection of $(\epsilon^*\lambda)^4$ with $\overline{D}$ is the degree of the closure of D in $\overline{\mathcal{M}_h}^{\mathrm{BB}}$ with respect to λ. Intersecting with $(\epsilon^*\lambda)^4$ defines a linear map given by the Baily–Borel degree $\deg : \mathrm{Pic}_{\mathbb{Q}}(\mathcal{M}_h) \longrightarrow \mathbb{Q}$. Theorem 2.2 gives us then that

$$\sum_{m,\mu} \left(H_{-m,\mu} \cdot (\epsilon^*\lambda)^4\right) q^m e_\mu \in \mathrm{Mod}^{\circ}_{\frac{7}{2},\Lambda_h}. \tag{23}$$

Further, by [Kud03, Theorem I] (see also [Kud03, Corollary 4.12] this is a multiple of the Eisenstein series $E_{\frac{7}{2},\Lambda_h}$ defined in (5). Now nefness implies

$$H_{-m,\mu} \cdot (\epsilon^*\lambda)^4 = -C \cdot c_{m,\mu}(E_{\frac{7}{2},\Lambda_d}) \quad \text{and} \quad (\epsilon^*\lambda)^5 = C \cdot c_{0,0}(E_{\frac{7}{2},\Lambda_d}), \tag{24}$$

where $c_{m,\mu} \in \left(\mathrm{Mod}^0_{\frac{7}{2},\Lambda_d}\right)^\vee$ is the (m,μ)-coefficient extraction functional, and C is a positive constant. By Proposition 4.4 we have $\frac{1}{4}H_{-1,0} \leq \frac{1}{2}\mathrm{Br}(\pi_{\mathrm{d}})$ and (24) gives us

$$\left(5\lambda - \frac{1}{2}\mathrm{Br}(\pi_d)\right) \cdot (\epsilon^*\lambda)^4 \leq \left(5\lambda - \frac{1}{4}H_{-1,0}\right) \cdot (\epsilon^*\lambda)^4 = C\left(5c_{0,0}\left(E_{\frac{7}{2},\Lambda_d}\right) + \frac{1}{4}c_{-1,0}\left(E_{\frac{7}{2},\Lambda_d}\right)\right). \tag{25}$$

There is a concrete formula [BK01] for the coefficients of the Fourier expansion of Eisenstein series. This has been implemented in SAGE by the fourth author [Wilb]. We exhibit the highest cases for which we obtain a negative intersection product. When $\gamma = 1$ and $d = 12$. In this case if we write $E_{\frac{7}{2},\Lambda_h} = \sum_{\mu \in D(\Lambda_h)} E_\mu(q) e_\mu$, then

$$E_0(q) = 1 - \frac{272}{13}q - \frac{1472}{13}q^2 - \frac{3390}{13}q^3 - \frac{8204}{13}q^4 + \mathrm{O}(q^5).$$

In particular from (25) we obtain

$$K_{Y_h} \cdot \eta^* \left(\epsilon^* \lambda\right)^4 \leq C \cdot \left(5 - \frac{272}{52}\right) < 0.$$

This gives us the theorem. When $d = 4t - 1$ and $\gamma = 2$, the highest degree for which we obtain uniruledness is $t = 12$. In this case the E_0 summand of $E_{\frac{7}{2},\Lambda_h}$ is

$$E_0(q) = 1 - \frac{1052352}{51911}q - \frac{5438160}{51911}q^2 - \frac{15409296}{51911}q^3 - \frac{907200}{1403}q^4 + \mathrm{O}(q^5)$$

and

$$K_{Y_h} \cdot \eta^* \left(\epsilon^* \lambda\right)^4 \leq C \cdot \left(5 - \frac{1052352}{4 \cdot 51911}\right) < 0.$$

which leads to the same result. Similarly, when $\gamma = 2$ and $d = 4t - 2$ with $t = 13$ the E_0 component of the Eisenstein series of weight $\frac{7}{2}$ corresponding to Λ_h is

$$E_0(q) = 1 - \frac{108}{5}q - \frac{620}{7}q^2 + \mathrm{O}(q^3),$$

which leads to the same result. The lower-degree cases are done in the same way. □

7.1. **Generalized Kummer case.** Let (X, L) be a primitively polarized hyperkähler $2n$-fold where X is deformation equivalent to a fiber of the addition map $A^{[n+1]} \longrightarrow A$ on an abelian surface. In this case the $(H^2(X,\mathbb{Z}), q_X)$ is isomorphic to $\Lambda = U^{\oplus 3} \oplus A_1(-(n+1))$ and the monodromy group [Mon16], [Mar23, Theorem 1.4] is:

$$\mathrm{Mon}^2\left(\Lambda\right) = \left\{ g \in \widehat{\mathrm{O}}^+\left(\Lambda\right) \middle| \chi(g) \cdot \det(g) = 1 \right\}, \tag{26}$$

where $\chi : \widehat{\mathrm{O}}^+\left(\Lambda\right) \longrightarrow \{\pm 1\}$ is the character defined by the action of $\widehat{\mathrm{O}}^+\left(\Lambda\right)$ on $D(\Lambda)$. Let $h = c_1(L) \in \Lambda$, with $\langle h, h \rangle = 2d$ and divisibility γ. Since $\widetilde{\mathrm{SO}}^+\left(\Lambda\right) \subset \mathrm{Mon}^2\left(\Lambda\right)$, up to monodromy one can always assume $h = \gamma(e + tf) - a\delta$ for appropriate t and a, where δ is the generator of $A_1(-(n+1))$.

For $\gamma = 1, 2$, the lattice Λ_h is in the form $U^{\oplus 2} \oplus Q_d$ (resp. Q_t with $d = 4t - (n+1)$) where

$$Q_d = \mathbb{Z}\ell + \mathbb{Z}\delta = \begin{pmatrix} -2d & 0 \\ 0 & -2(n+1) \end{pmatrix} \quad \text{and} \quad Q_t = \mathbb{Z}u + \mathbb{Z}v = \begin{pmatrix} -2t & (n+1) \\ (n+1) & -2(n+1) \end{pmatrix}.$$

Lemma 7.3. *For $\gamma = 1, 2$, the polarized monodromy group* $\mathrm{Mon}^2(\Lambda, h) \subset \mathrm{O}^+(\Lambda_h)$ *is a double extension of* $\widetilde{\mathrm{SO}}^+(\Lambda_h)$. *More precisely,*

$$\mathrm{Mon}^2(\Lambda, h) = \langle \widetilde{\mathrm{SO}}^+(\Lambda_h), \sigma_\kappa \rangle, \quad \textit{where} \quad \kappa = \begin{cases} \delta & \textit{if } \gamma = 1, \\ v & \textit{if } \gamma = 2. \end{cases}$$

For $\gamma \geq 3$ there is equality $\mathrm{Mon}^2(\Lambda, h) = \widetilde{\mathrm{SO}}^+(\Lambda_h)$.

Proof. Observe that, for any $g \in O(\Lambda, h)$, we have $\det(g) = \det(g \mid_{\Lambda_h})$. Then the statement is essentially [ABL24, Lemma 3.7]. For $\gamma = 1, 2$ we also need to prove that $\sigma_\kappa \in \mathrm{Mon}^2(\Lambda)$ via the restriction: since $\det(\sigma_\kappa) = -1$, this is equivalent to prove that, if we see κ as an element of Λ, we have $-\sigma_\kappa \in \widetilde{\mathrm{O}}^+(\Lambda)$ i.e. $\chi(\sigma_\kappa) = -1$. Since $\kappa = 3(\gamma - 1)f - \delta$, this can be checked via an explicit computation. □

For $\gamma = 1, 2$, the period domain $\mathcal{M}^\gamma_{\mathrm{Kum_n,2d}} = \mathcal{D}_{\Lambda_h}/\mathrm{Mon}^2(\Lambda, h)$ is a partial compactification of the moduli space of hyperkähler $2n$-folds of generalized Kummer type with a primitive polarization of degree $2d$ and divisibility γ. It is always irreducible [Ono22] and never empty for $\gamma = 1$ (the *split case*). When $\gamma = 2$ it is non-empty only for $d \equiv -(n+1) \pmod 4$.

Lemma 7.4. *For $d = 1$ and $\gamma = 1, 2$, one has*

$$\left\langle \mathrm{Mon}^2(\Lambda, h), -\mathrm{Id} \right\rangle = \widehat{\mathrm{O}}^+(\Lambda_h), \tag{27}$$

or equivalently $\mathrm{PMon}^2(\Lambda, h) = \mathrm{P}\widehat{\mathrm{O}}^+(\Lambda_h) = \mathrm{P}\widetilde{\mathrm{O}}^+(\Lambda_h)$.

Proof. Note that, under our hypothesis, $-\sigma_\kappa \in \widetilde{\mathrm{O}}^+(\Lambda_h)$. For $\gamma = 1$, this holds since $\ell_* = -\ell_*$. For $\gamma = 2$, observe that $|D(\Lambda_h)| = d \cdot (n+1) = n+1$, hence $D(\Lambda_h) = \langle \kappa_* \rangle$ since $\kappa = v$ is primitive with divisibility $n+1$; clearly $\sigma_\kappa(\kappa_*) = -\kappa_*$. Now we prove (27) under the more general hypothesis that $-\sigma_\kappa \in \widetilde{\mathrm{O}}^+(\Lambda_2)$ i.e. $\chi(\sigma_\kappa) = -1$.

We can write $\widehat{\mathrm{O}}^+(\Lambda_h) = \bigcup_{i,j \in \{-1,+1\}} M_{i,j}$, where $M_{i,j}$ is the set of isometries $g \in \widehat{\mathrm{O}}^+(\Lambda_2)$ such that $(\chi(g), \det(g)) = (i, j)$. Clearly $\widetilde{\mathrm{SO}}^+(\Lambda_h) = M_{+1,+1}$ and, under our hypothesis, $\sigma_\kappa \cdot \widehat{\mathrm{SO}}^+(\Lambda_h) = M_{-1,-1}$. By Lemma 7.3 then $\mathrm{Mon}^2(\Lambda, h) = M_{+1,+1} \cup M_{-1,-1}$. Now $-\mathrm{Id} \in M_{-1,1}$, since Λ_h has even rank, hence $-\mathrm{Id} \cdot M_{i,j} = M_{-i,j}$ and (27) follows. □

Theorem 7.5. *The moduli spaces $\mathcal{M}^1_{\mathrm{Kum}_n,2}$ and $\mathcal{M}^2_{\mathrm{Kum}_n,2}$ of hyperkähler $2n$-folds of generalized Kummer type with polarization of degree 2 and divisibility $\gamma = 1, 2$ are uniruled in the following cases:*

(i) *when $\gamma = 1$ for $n \leq 15$ and $n = 17, 20$,*
(ii) *when $\gamma = 2$ for $t \leq 11$ and $t = 13, 15, 17, 19$, where $n = 4t - 2$.*

Proof. We show that $\mathcal{D}_{\Lambda_h}/\widetilde{\mathrm{O}}^+(\Lambda_h)$ is uniruled and by Lemma 7.4 we conclude uniruledness for $\mathcal{M}^\gamma_{\mathrm{Kum}_2,2}$. As in the proof of Theorem 7.2, from Proposition 4.4 we have that the intersection of $(\epsilon^*\lambda)^3$ with the canonical bundle K_Y on a smooth model of a toroidal compactification is bounded from above by

$$K_Y \cdot (\epsilon^*\lambda)^3 \leq \left(4\lambda - \frac{1}{4}\left(H_{-1,0}\right)\right) \cdot (\epsilon^*\lambda)^3 = 4c_{0,0}(E) + \frac{1}{4}c_{-1,0}(E), \tag{28}$$

where $E = E_{3,\Lambda_h}$ is the Eisenstein series corresponding to Λ_h. Again, we exhibit only one case. If $E = \sum_{\mu \in D(\Lambda_h)} E_\mu(q)e_\mu$, one computes [Wilb]:

$$E_0(q) = \begin{cases} 1 - \frac{4250}{263}q - \frac{12600}{263}q^2 + O(q^3) & \text{if } n = 20 \text{ and } \gamma = 1 \\ 1 - \frac{130}{7}q - \frac{288}{7}q^2 + O(q^3) & \text{if } n = 4t-2 \text{ with } t = 19 \text{ and } \gamma = 2. \end{cases}$$

From (28) we obtain $K_Y \cdot (\epsilon^*\lambda)^3 < 0$. Since $(\epsilon^*\lambda)^3$ is nef, the canonical class K_Y in both cases sits outside the respective pseudo-effective cones. Uniruledness follows from [BDPP13]. □

We remark here that (see Lemma 7.4) the modular variety

$$\mathcal{M}^2_{\mathrm{Kum}_2,2} = \mathcal{D}_{\Lambda_h} / \widetilde{\mathrm{O}}^+\left(\Lambda_h\right),$$

where $\Lambda_h = U^{\oplus 2} \oplus A_2(-1)$, is known to be rational [WW21, Theorem 5.4]. More concretely, there is a finite union of Heegner divisors $\mathcal{H}$, see [WW21, Equation 5.8], such that the algebra of meromorphic modular forms $\mathrm{M}^!_*\left(\widetilde{\mathrm{O}}^+(\Lambda_h), \mathcal{H}\right)$, that is, meromorphic sections of $\lambda^{\otimes k}$ with $k \in \mathbb{Z}$ and poles supported along $\mathcal{H}$ is finitely generated by forms of positive weight. By work of Looijenga [Loo03] the projective variety $\widehat{X} = \mathrm{Proj}\left(\oplus_{k\geq 0}\mathrm{M}^!_k\left(\widetilde{\mathrm{O}}^+, \mathcal{H}\right)\right)$ is a compactification of $\mathcal{D}/\widetilde{\mathrm{O}}^+ - \mathcal{H}$ that interpolates between the Baily–Borel and toroidal compactifications. When the generators are relation-free, as it is shown in [WW21] for $\Lambda_h = U^{\oplus 2} \oplus A_2(-1)$, the resulting ring is a polynomial algebra with generators of mixed weights. In this case $\widehat{X}$ is a weighted projective space, in particular rational. The same holds for some of the first OG6 cases. Indeed if $\Lambda_h = U^{\oplus 2} \oplus A_1(-1)^{\oplus 3}$ or $\Lambda_h = U^{\oplus 2} \oplus A_1(-1)^{\oplus 1} \oplus A_2(-1)$, then [WW21, Theorem 5.4] implies that the resulting modular varieties $\mathcal{D}_{\Lambda_h}/\widetilde{\mathrm{O}}^+(\Lambda_h)$ are also rational. We summarize the results relevant for this paper:

Theorem 7.6 (Theorem 5.4 in [WW21]). *The moduli space $\mathcal{M}^2_{\mathrm{Kum}_2,2}$ is rational and the moduli spaces $\mathcal{M}^2_{\mathrm{OG6},6}$ and $\mathcal{M}^1_{\mathrm{OG6},2}$ are unirational.*

Proof. This is an immediate consequence of [WW21, Theorem 5.4] together with Lemmas 7.1 and 7.4. □

We observe that the strategy in Theorem 7.5 fails for $\gamma \geq 3$. In this case, a nef curve intersecting the canonical class negatively would have to intersect the boundary of a toroidal compactification. The reason for this is that the canonical class sits always in the interior of the NL-cone, even further, it is the restriction of an ample class on the Baily–Borel model.

Proposition 7.7. *For $\gamma = 3, 6$, when non-empty, the canonical class of every component $\mathcal{M}$ of the moduli space $\mathcal{M}^{\gamma}_{\mathrm{Kum}_2,2d}$ is given by*

$$K_{\mathcal{M}} = 4\lambda.$$

In particular, it lies in the interior of the NL-cone and it has positive intersection with any complete curve not intersecting the boundary of a toroidal compactification.

Proof. By Lemma 7.3, the branch divisor of the modular projection $\pi : \mathcal{D}_{\Lambda_h} \longrightarrow \mathcal{D}_{\Lambda_h}/\mathrm{Mon}^2(\Lambda, h)$ is trivial, since both σ_ρ and $-\sigma_\rho$ have negative determinant on a lattice of even rank, see Equation (14). □

Finally, we describe the structure of the NL-cone for some of these moduli spaces:

Theorem 7.8. *The NL-cones in terms of the minimal set of generating rays for the moduli spaces $\mathcal{M}^1_{\mathrm{Kum}_2,2}, \mathcal{M}^2_{\mathrm{Kum}_2,2}$ are given by:*

$$\mathrm{Eff}^{NL}\left(\mathcal{M}^1_{\mathrm{Kum}_2,2}\right) = \left\langle P_{-\frac{1}{12},\delta_*}, P_{-\frac{1}{4},\ell_*}\right\rangle_{\mathbb{Q}_{\geq 0}} \quad \textit{and} \quad \mathrm{Eff}^{NL}\left(\mathcal{M}^2_{\mathrm{Kum}_2,2}\right) = \mathbb{Q}_{\geq 0} P_{-\frac{1}{3},\frac{v}{3}}.$$

Further, the moduli space $\left(\mathcal{M}^2_{\mathrm{Kum}_2,2}\right)^\circ$ parameterizing polarized hyperkähler fourfolds with polarization of degree 2 and divisibility 2 is quasi-affine.

Proof. The structure of the cones follows from Theorem 1.1 and the results of Section 3 together with [Wila]. Recall that the period map [Ver13]

$$\left(\mathcal{M}^{\gamma}_{\mathrm{Kum}_2,2d}\right)^\circ \longrightarrow \mathcal{D}_{\Lambda_h}/\mathrm{Mon}(\Lambda, h) = \mathcal{M}^{\gamma}_{\mathrm{Kum}_2,2d} \tag{29}$$

is an open embedding. In particular, if the complement $\mathcal{M}^{\gamma}_{\mathrm{Kum}_2,2d} - \left(\mathcal{M}^{\gamma}_{\mathrm{Kum}_2,2d}\right)^\circ$ contains an ample divisor, then $\left(\mathcal{M}^{\gamma}_{\mathrm{Kum}_2,2d}\right)^\circ$ is quasi-affine. When $\gamma = 2$ and $d = 1$, it is enough to show that the complement contains a divisional component of NL-type. Indeed, if a primitive Heegner divisor P_ρ is contained in the complement of the image of (29), then P_ρ must be a positive rational multiple of λ, in particular, ample. Then, $\left(\mathcal{M}^2_{\mathrm{Kum}_2,2}\right)^\circ$ is an open in the complement of a hyperplane in $\overline{\mathcal{D}/\widetilde{\mathrm{O}}^+}^{BB} \subset \mathbb{P}^N$, i.e. quasi-affine.

Recall that if (X, H) is a polarized hyperkähler fourfold of Kum_2-type, then

$$\left(H^2(X, \mathbb{Z}), q_X\right) \cong \Lambda$$

with $\Lambda = U^{\oplus 3} \oplus A_1(-3)$. We call δ the generator of the last factor and $h = c_1(H)$. By [Yos16], see also [MTW18, Page 452], an ample class h cannot lie in the orthogonal complement in $H^{1,1}(X, \mathbb{R})$ of classes $\rho \in \mathrm{NS}(X)$ whose square is -6 and divisibility in $H^2(X, \mathbb{Z})$ is $2, 3$ or 6. In particular, if such a class is orthogonal to h, then D_ρ defines a hyperplane in $\mathcal{D}_{\Lambda_h}$ and the image of the period map misses the corresponding divisor P_ρ. Singling out classes in $H^{1,1}(X, \mathbb{Z})$ whose orthogonal complements give the chamber decomposition of the positive cone $C(X) \subset H^{1,1}(X, \mathbb{R})$ is a general method to describe the complement of the image of the period map, see for instance [DM19, Theorem 6.1]. In the second case of the theorem, it is enough to show that there exists an integral class

$\rho \in \Lambda_h$ of square $\langle \rho, \rho \rangle = -6$ and divisibility in Λ given by $\mathrm{div}_\Lambda(\rho) \in \{2, 3, 6\}$. Since $\widetilde{\mathrm{SO}}^+(\Lambda) \subset \mathrm{Mon}^2(\Lambda)$, one can assume $h = 2(e+f) - \delta$, and taking $\rho = 3f - \delta$ one has the desired property. In this case the divisibility in Λ is 3, and the missed primitive Heegner divisor in $\mathcal{M}^2_{\mathrm{Kum}_2,2}$ is $P_\rho = P_{-\frac{1}{3},\frac{v}{3}}$. □

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8. Appendix

Here we list various computations of generating rays for NL-cones, branch divisors, and NL-positivity of the canonical class for moduli spaces of hyperkähler varieties.

Table 1: Generating rays of the NL-cone of $\mathcal{F}_{2d}$

d	generating rays of $\mathrm{Eff}^{NL}(\mathcal{F}_{2d})$	# gen. rays	$\dim_{\mathbb{Q}} \mathrm{Pic}(\mathcal{F}_{2d})$
1	$P_{-1,0}, P_{-\frac{1}{4},\ell_*}$	2	2
2	$P_{-1,0}, P_{-\frac{1}{8},\ell_*}, P_{-\frac{1}{2},2\ell_*}$	3	3
3	$P_{-1,0}, P_{-\frac{1}{12},\ell_*}, P_{-\frac{1}{3},2\ell_*}, P_{-\frac{3}{4},3\ell_*}$	4	4
4	$P_{-1,0}, P_{-\frac{1}{16},\ell_*}, P_{-\frac{1}{4},2\ell_*}, P_{-\frac{9}{16},3\ell_*}, P_{-1,4\ell_*}$	5	4
5	$P_{-1,0}, P_{-\frac{1}{20},\ell_*}, P_{-\frac{1}{5},2\ell_*}, P_{-\frac{9}{20},3\ell_*}, P_{-\frac{4}{5},4\ell_*}, P_{-\frac{1}{4},5\ell_*}$	6	6
6	$P_{-1,0}, P_{-\frac{1}{24},\ell_*}, P_{-\frac{1}{6},2\ell_*}, P_{-\frac{3}{8},3\ell_*}, P_{-\frac{2}{3},4\ell_*}, P_{-\frac{1}{24},5\ell_*}$ $P_{-\frac{1}{2},6\ell_*}$	7	7
7	$P_{-1,0}, P_{-\frac{1}{28},\ell_*}, P_{-\frac{1}{7},2\ell_*}, P_{-\frac{9}{28},3\ell_*}, P_{-\frac{4}{7},4\ell_*}, P_{-\frac{25}{28},5\ell_*}$ $P_{-\frac{2}{7},6\ell_*}, P_{-\frac{3}{4},7\ell_*}$	8	7
8	$P_{-1,0}, P_{-\frac{1}{32},\ell_*}, P_{-\frac{33}{32},\ell_*}, P_{-\frac{1}{8},2\ell_*}, P_{-\frac{9}{32},3\ell_*}, P_{-\frac{1}{2},4\ell_*}$ $P_{-\frac{25}{32},5\ell_*}, P_{-\frac{1}{8},6\ell_*}, P_{-\frac{17}{32},7\ell_*}, P_{-1,8\ell_*}$	10	8
9	$P_{-1,0}, P_{-\frac{1}{36},\ell_*}, P_{-\frac{37}{36},\ell_*}, P_{-\frac{1}{9},2\ell_*}, P_{-\frac{10}{9},2\ell_*}, P_{-\frac{1}{4},3\ell_*}$ $P_{-\frac{4}{9},4\ell_*}, P_{-\frac{25}{36},5\ell_*}, P_{-1,6\ell_*}, P_{-\frac{13}{36},7\ell_*}, P_{-\frac{7}{9},8\ell_*}, P_{-\frac{1}{4},9\ell_*},$ $P_{-\frac{5}{4},9\ell_*}$	13	9
10	$P_{-1,0}, P_{-\frac{1}{40},\ell_*}, P_{-\frac{1}{10},2\ell_*}, P_{-\frac{9}{40},3\ell_*}, P_{-\frac{2}{5},4\ell_*}, P_{-\frac{5}{8},5\ell_*}$ $P_{-\frac{9}{10},6\ell_*}, P_{-\frac{9}{40},7\ell_*}, P_{-\frac{3}{5},8\ell_*}, P_{-\frac{1}{40},9\ell_*}, P_{-\frac{1}{2},10\ell_*}$	11	10
11	$P_{-1,0}, P_{-\frac{1}{44},\ell_*}, P_{-\frac{45}{44},\ell_*}, P_{-\frac{1}{11},2\ell_*}, P_{-\frac{12}{11},2\ell_*}, P_{-\frac{9}{44},3\ell_*}$ $P_{-\frac{53}{44},3\ell_*}, P_{-\frac{4}{11},4\ell_*}, P_{-\frac{25}{44},5\ell_*}, P_{-\frac{9}{11},6\ell_*}, P_{-\frac{5}{44},7\ell_*}, P_{-\frac{49}{44},7\ell_*},$ $P_{-\frac{5}{11},8\ell_*}, P_{-\frac{37}{44},9\ell_*}, P_{-\frac{3}{11},10\ell_*}, P_{-\frac{3}{4},11\ell_*}$	16	11
12	$P_{-1,0}, P_{-\frac{1}{48},\ell_*}, P_{-\frac{49}{48},\ell_*}, P_{-\frac{1}{12},2\ell_*}, P_{-\frac{3}{16},3\ell_*}, P_{-\frac{1}{3},4\ell_*}$ $P_{-\frac{25}{48},5\ell_*}, P_{-\frac{3}{4},6\ell_*}, P_{-\frac{1}{48},7\ell_*}, P_{-\frac{49}{48},7\ell_*}, P_{-\frac{1}{3},8\ell_*}, P_{-\frac{11}{16},9\ell_*},$ $P_{-\frac{1}{12},10\ell_*}, P_{-\frac{25}{48},11\ell_*}, P_{-1,12\ell_*}$	15	12
13	$P_{-1,0}, P_{-\frac{1}{52},\ell_*}, P_{-\frac{53}{52},\ell_*}, P_{-\frac{1}{13},2\ell_*}, P_{-\frac{9}{52},3\ell_*}, P_{-\frac{4}{13},4\ell_*}$ $P_{-\frac{25}{52},5\ell_*}, P_{-\frac{9}{13},6\ell_*}, P_{-\frac{49}{52},7\ell_*}, P_{-\frac{3}{13},8\ell_*}, P_{-\frac{29}{52},9\ell_*}, P_{-\frac{12}{13},10\ell_*},$ $P_{-\frac{17}{52},11\ell_*}, P_{-\frac{10}{13},12\ell_*}, P_{-\frac{1}{4},13\ell_*}, P_{-\frac{5}{4},13\ell_*}$	16	12
14	$P_{-1,0}, P_{-\frac{1}{56},\ell_*}, P_{-\frac{57}{56},\ell_*}, P_{-\frac{1}{14},2\ell_*}, P_{-\frac{9}{56},3\ell_*}, P_{-\frac{2}{7},4\ell_*}$ $P_{-\frac{25}{56},5\ell_*}, P_{-\frac{9}{14},6\ell_*}, P_{-\frac{7}{8},7\ell_*}, P_{-\frac{1}{7},8\ell_*}, P_{-\frac{25}{56},9\ell_*}, P_{-\frac{11}{14},10\ell_*},$ $P_{-\frac{9}{56},11\ell_*}, P_{-\frac{4}{7},12\ell_*}, P_{-\frac{1}{56},13\ell_*}, P_{-\frac{57}{56},13\ell_*}, P_{-\frac{1}{2},14\ell_*}, P_{-\frac{3}{2},14\ell_*}$	18	14
15	$P_{-1,0}, P_{-\frac{1}{60},\ell_*}, P_{-\frac{61}{60},\ell_*}, P_{-\frac{1}{15},2\ell_*}, P_{-\frac{16}{15},2\ell_*}, P_{-\frac{3}{20},3\ell_*}$ $P_{-\frac{4}{15},4\ell_*}, P_{-\frac{5}{12},5\ell_*}, P_{-\frac{3}{5},6\ell_*}, P_{-\frac{49}{60},7\ell_*}, P_{-\frac{1}{15},8\ell_*}, P_{-\frac{16}{15},8\ell_*},$ $P_{-\frac{7}{20},9\ell_*}, P_{-\frac{2}{3},10\ell_*}, P_{-\frac{1}{60},11\ell_*}, P_{-\frac{61}{60},11\ell_*}, P_{-\frac{2}{5},12\ell_*}, P_{-\frac{49}{60},13\ell_*},$ $P_{-\frac{4}{15},14\ell_*}, P_{-\frac{3}{4},15\ell_*}$	20	15

16	$P_{-1,0}, P_{-\frac{1}{64},\ell_*}, P_{-\frac{65}{64},\ell_*}, P_{-\frac{1}{16},2\ell_*}, P_{-\frac{17}{16},2\ell_*}, P_{-\frac{9}{64},3\ell_*}$ $P_{-\frac{1}{4},4\ell_*}, P_{-\frac{25}{64},5\ell_*}, P_{-\frac{9}{16},6\ell_*}, P_{-\frac{49}{64},7\ell_*}, P_{-1,8\ell_*}, P_{-\frac{17}{64},9\ell_*},$ $P_{-\frac{9}{16},10\ell_*}, P_{-\frac{57}{64},11\ell_*}, P_{-\frac{1}{4},12\ell_*}, P_{-\frac{41}{64},13\ell_*}, P_{-\frac{1}{16},14\ell_*}, P_{-\frac{17}{16},14\ell_*},$ $P_{-\frac{33}{64},15\ell_*}, P_{-1,16\ell_*}$	20	14
17	$P_{-1,0}, P_{-\frac{1}{68},\ell_*}, P_{-\frac{69}{68},\ell_*}, P_{-\frac{1}{17},2\ell_*}, P_{-\frac{18}{17},2\ell_*}, P_{-\frac{9}{68},3\ell_*}$ $P_{-\frac{77}{68},3\ell_*}, P_{-\frac{4}{17},4\ell_*}, P_{-\frac{25}{68},5\ell_*}, P_{-\frac{9}{17},6\ell_*}, P_{-\frac{49}{68},7\ell_*}, P_{-\frac{16}{17},8\ell_*},$ $P_{-\frac{13}{68},9\ell_*}, P_{-\frac{8}{17},10\ell_*}, P_{-\frac{53}{68},11\ell_*}, P_{-\frac{2}{17},12\ell_*}, P_{-\frac{19}{17},12\ell_*}, P_{-\frac{33}{68},13\ell_*},$ $P_{-\frac{15}{17},14\ell_*}, P_{-\frac{21}{68},15\ell_*}, P_{-\frac{13}{17},16\ell_*}, P_{-\frac{1}{4},17\ell_*}, P_{-\frac{5}{4},17\ell_*}$	23	16
18	$P_{-1,0}, P_{-\frac{1}{72},\ell_*}, P_{-\frac{73}{72},\ell_*}, P_{-\frac{1}{18},2\ell_*}, P_{-\frac{19}{18},2\ell_*}, P_{-\frac{1}{8},3\ell_*}$ $P_{-\frac{2}{9},4\ell_*}, P_{-\frac{11}{9},4\ell_*}, P_{-\frac{25}{72},5\ell_*}, P_{-\frac{1}{2},6\ell_*}, P_{-\frac{49}{72},7\ell_*}, P_{-\frac{8}{9},8\ell_*},$ $P_{-\frac{1}{8},9\ell_*}, P_{-\frac{9}{8},9\ell_*}, P_{-\frac{7}{18},10\ell_*}, P_{-\frac{49}{72},11\ell_*}, P_{-1,12\ell_*}, P_{-\frac{25}{72},13\ell_*},$ $P_{-\frac{13}{18},14\ell_*}, P_{-\frac{1}{8},15\ell_*}, P_{-\frac{5}{9},16\ell_*}, P_{-\frac{1}{72},17\ell_*}, P_{-\frac{73}{72},17\ell_*}, P_{-\frac{1}{2},18\ell_*},$ $P_{-\frac{3}{2},18\ell_*}$	25	17
19	$P_{-1,0}, P_{-\frac{1}{76},\ell_*}, P_{-\frac{77}{76},\ell_*}, P_{-\frac{1}{19},2\ell_*}, P_{-\frac{20}{19},2\ell_*}, P_{-\frac{9}{76},3\ell_*}$ $P_{-\frac{85}{76},3\ell_*}, P_{-\frac{4}{19},4\ell_*}, P_{-\frac{23}{19},4\ell_*}, P_{-\frac{25}{76},5\ell_*}, P_{-\frac{9}{19},6\ell_*}, P_{-\frac{49}{76},7\ell_*},$ $P_{-\frac{16}{19},8\ell_*}, P_{-\frac{5}{76},9\ell_*}, P_{-\frac{81}{76},9\ell_*}, P_{-\frac{6}{19},10\ell_*}, P_{-\frac{45}{76},11\ell_*}, P_{-\frac{17}{19},12\ell_*},$ $P_{-\frac{17}{76},13\ell_*}, P_{-\frac{93}{76},13\ell_*}, P_{-\frac{11}{19},14\ell_*}, P_{-\frac{73}{76},15\ell_*}, P_{-\frac{7}{19},16\ell_*}, P_{-\frac{61}{76},17\ell_*},$ $P_{-\frac{5}{19},18\ell_*}, P_{-\frac{24}{19},18\ell_*}, P_{-\frac{3}{4},19\ell_*}$	27	17
20	$P_{-1,0}, P_{-2,0}, P_{-\frac{1}{80},\ell_*}, P_{-\frac{81}{80},\ell_*}, P_{-\frac{1}{20},2\ell_*}, P_{-\frac{21}{20},2\ell_*}$ $P_{-\frac{9}{80},3\ell_*}, P_{-\frac{89}{80},3\ell_*}, P_{-\frac{1}{5},4\ell_*}, P_{-\frac{5}{16},5\ell_*}, P_{-\frac{9}{20},6\ell_*}, P_{-\frac{49}{80},7\ell_*},$ $P_{-\frac{4}{5},8\ell_*}, P_{-\frac{1}{80},9\ell_*}, P_{-\frac{81}{80},9\ell_*}, P_{-\frac{1}{4},10\ell_*}, P_{-\frac{41}{80},11\ell_*}, P_{-\frac{4}{5},12\ell_*},$ $P_{-\frac{9}{80},13\ell_*}, P_{-\frac{89}{80},13\ell_*}, P_{-\frac{9}{20},14\ell_*}, P_{-\frac{13}{16},15\ell_*}, P_{-\frac{1}{5},16\ell_*}, P_{-\frac{49}{80},17\ell_*},$ $P_{-\frac{1}{20},18\ell_*}, P_{-\frac{21}{20},18\ell_*}, P_{-\frac{41}{80},19\ell_*}, P_{-1,20\ell_*}$	28	19

TABLE 2. Irreducible components of $\phi_r^*P_{-1,0}$ and $\phi_r^*P_{-\frac{1}{4},\ell_*}$ under $\phi_r\colon F_{2dr^2} \to \mathcal{F}_2$

r	Components of $\phi_r^*P_{-1,0}$	Components of $\phi_r^*P_{-\frac{1}{4},\ell_*}$
2	$P_{-1,0}, P_{-1,4\ell'_*}$	$P_{-\frac{1}{4},2\ell'_*}$
3	$P_{-1,0}, P_{-\frac{1}{9},2\ell'_*}, P_{-1,6\ell'_*}$	$P_{-\frac{1}{4},3\ell'_*}, P_{-\frac{1}{4},9\ell'_*}$
4	$P_{-1,0}, P_{-1,8\ell'_*}, P_{-1,16\ell'_*}$	$P_{-\frac{1}{4},4\ell'_*}, P_{-\frac{1}{4},12\ell'_*}$
5	$P_{-1,0}, P_{-1,10\ell'_*}, P_{-1,20\ell'_*}, P_{-\frac{1}{25},2\ell'_*}$	$P_{-\frac{1}{4},5\ell'_*}, P_{-\frac{1}{4},15\ell'_*}, P_{-\frac{1}{4},25\ell'_*}$
6	$P_{-1,0}, P_{-1,12\ell'_*}, P_{-1,24\ell'_*}, P_{-1,36\ell'_*}$	$P_{-\frac{1}{4},6\ell'_*}, P_{-\frac{1}{4},18\ell'_*}, P_{-\frac{1}{4},30\ell'_*}$
7	$P_{-1,0}, P_{-\frac{1}{49},2\ell'_*}, P_{-1,14\ell'_*}, P_{-1,28\ell'_*}, P_{-1,42\ell'_*},$	$P_{-\frac{1}{4},7\ell'_*}, P_{-\frac{1}{4},21\ell'_*}, P_{-\frac{1}{4},35\ell'_*}, P_{-\frac{1}{4},49\ell'_*}$
8	$P_{-1,0}, P_{-1,16\ell'_*}, P_{-\frac{1}{9},4\ell'_*}, P_{-1,32\ell'_*}, P_{-1,48\ell'_*}, P_{-1,64\ell'_*}$	$P_{-\frac{1}{4},8\ell'_*}, P_{-\frac{1}{4},24\ell'_*}, P_{-\frac{1}{4},40\ell'_*}, P_{-\frac{1}{4},56\ell'_*}$

TABLE 3. Ramification divisor of $\pi_d : \mathcal{D} \longrightarrow \mathcal{F}_{2d}$.

d	in terms of P's	in terms of H's
1	$\frac{1}{2}\left(P_{-1,0} + P_{-\frac{1}{4},\ell_*} + P_{-\frac{1}{4},\delta_*}\right)$	$\frac{1}{2}H_{-1,0}$
2	$\frac{1}{2}\left(P_{-1,0} + P_{-\frac{1}{4},\delta_*} + P_{-\frac{1}{2},2\ell_*}\right) + P_{-\frac{1}{8},\ell_*}$	$\frac{1}{2}\left(H_{-1,0} + H_{-\frac{1}{2},2\ell_*}\right)$
3	$\frac{1}{2}\left(P_{-1,0} + P_{-\frac{1}{4},\delta_*}\right) + P_{-\frac{1}{3},2\ell_*} + P_{-\frac{1}{12},\ell_*}$	$\frac{1}{2}H_{-1,0} + H_{-\frac{1}{3},2\ell_*}$
4	$\frac{1}{2}\left(P_{-1,0} + P_{-\frac{1}{4},\delta_*} + P_{-\frac{1}{4},4\ell_*+\delta_*}\right) + P_{-\frac{1}{4},2\ell_*} + P_{-\frac{1}{16},\ell_*}$	$\frac{1}{2}H_{-1,0} + H_{-\frac{1}{4},2\ell_*}$
5	$\frac{1}{2}\left(P_{-1,0} + P_{-\frac{1}{4},\delta_*} + P_{-\frac{1}{4},5\ell_*}\right) + P_{-\frac{1}{5},2\ell_*} + P_{-\frac{1}{20},\ell_*} + P_{-\frac{1}{20},4\ell_*+\delta_*}$	$\frac{1}{2}H_{-1,0} + H_{-\frac{1}{5},2\ell_*}$
6	$\frac{1}{2}\left(P_{-1,0} + P_{-\frac{1}{4},\delta_*}\right) + P_{-\frac{1}{6},2\ell_*} + P_{-\frac{1}{24},\ell_*} + P_{-\frac{1}{24},5\ell_*}$	$\frac{1}{2}H_{-1,0} + H_{-\frac{1}{6},2\ell_*}$

TABLE 4. Generating rays of the NL-cone of $\mathcal{M}^1_{\mathrm{K3}^{[2]},2d}$ with $d \leq 5$

d	generating rays of $\mathrm{Eff}^{NL}\left(\mathcal{M}^1_{\mathrm{K3}^{[2]},2d}\right)$	Picard rank	position of K
1	$P_{-1,0}, P_{-\frac{1}{4},\ell_*}, P_{-\frac{1}{4},\delta_*}, P_{-\frac{1}{2},\ell_*+\delta_*}$	4	out
2	$P_{-1,0}, P_{-\frac{1}{8},\ell_*}, P_{-\frac{9}{8},\ell_*}, P_{-\frac{1}{4},\delta_*}, P_{-\frac{5}{4},\delta_*}, P_{-\frac{3}{8},\ell_*+\delta_*}, P_{-\frac{1}{2},2\ell_*}, P_{-\frac{3}{4},2\ell_*+\delta_*}$	6	in
3	$P_{-1,0}, P_{-\frac{1}{12},\ell_*}, P_{-\frac{13}{12},\ell_*}, P_{-\frac{1}{4},\delta_*}, P_{-\frac{1}{3},\ell_*+\delta_*}, P_{-\frac{1}{3},2\ell_*}, P_{-\frac{7}{12},2\ell_*+\delta_*}, P_{-\frac{3}{4},3\ell_*}, P_{-1,3\ell_*+\delta_*}$	7	in
4	$P_{-1,0}, P_{-\frac{1}{16},\ell_*}, P_{-\frac{17}{16},\ell_*}, P_{-\frac{1}{4},\delta_*}, P_{-\frac{5}{16},\ell_*+\delta_*}, P_{-\frac{1}{4},2\ell_*}, P_{-\frac{5}{4},2\ell_*}, P_{-\frac{1}{2},2\ell_*+\delta_*}, P_{-\frac{9}{16},3\ell_*}, P_{-\frac{13}{16},3\ell_*+\delta_*}, P_{-1,4\ell_*}, P_{-\frac{1}{4},4\ell_*+\delta_*}$	9	in
5	$P_{-1,0}, P_{-\frac{1}{20},\ell_*}, P_{-\frac{21}{20},\ell_*}, P_{-\frac{1}{4},\delta_*}, P_{-\frac{3}{10},\ell_*+\delta_*}, P_{-\frac{1}{5},2\ell_*}, P_{-\frac{6}{5},2\ell_*}, P_{-\frac{9}{20},2\ell_*+\delta_*}, P_{-\frac{9}{20},3\ell_*}, P_{-\frac{7}{10},3\ell_*+\delta_*}, P_{-\frac{4}{5},4\ell_*}, P_{-\frac{1}{20},4\ell_*+\delta_*}, P_{-\frac{21}{20},4\ell_*+\delta_*},, P_{-\frac{1}{4},5\ell_*}, P_{-\frac{5}{4},5\ell_*}, P_{-\frac{1}{2},5\ell_*+\delta_*}$	12	in

TABLE 5. Generating rays of the NL-cone of $\mathcal{M}^2_{\mathrm{K3}^{[2]},8t-2}$ with $t \leq 5$

t	generating rays of $\mathrm{Eff}^{NL}\left(\mathcal{M}^2_{\mathrm{K3}^{[2]},8t-2}\right)$	Picard rank	position of K
1	$P_{-1,0}, P_{-\frac{1}{3},\frac{(2u+v)_*}{3}}$	2	out
2	$P_{-1,0}, P_{-\frac{1}{7},\frac{(2u+v)_*}{7}}, P_{-\frac{4}{7},\frac{(4u+2v)_*}{7}}, P_{-\frac{2}{7},\frac{(6u+3v)_*}{7}}$	4	in
3	$P_{-1,0}, P_{-\frac{3}{11},\frac{(u+6v)_*}{11}}, P_{-\frac{1}{11},\frac{(2u+v)_*}{11}}, P_{-\frac{12}{11},\frac{(2u+v)_*}{11}}$, $P_{-\frac{4}{11},\frac{(4u+2v)_*}{11}}, P_{-\frac{9}{11},\frac{(6u+3v)_*}{11}}, P_{-\frac{5}{11},\frac{(8u+4v)_*}{11}}$	6	in
4	$P_{-1,0}, P_{-\frac{2}{3},\frac{u+2v}{3}}, P_{-\frac{4}{15},\frac{u+8v}{15}}, P_{-\frac{1}{15},\frac{2u+v}{15}}$, $P_{-\frac{16}{15},\frac{2u+v}{15}}, P_{-\frac{2}{5},\frac{3u+2v}{5}}, P_{-\frac{3}{5},\frac{3u+4v}{5}}, P_{-\frac{4}{15},\frac{4u+2v}{15}}$, $P_{-\frac{1}{15},\frac{7u+11v}{15}}, P_{-\frac{16}{15},\frac{7u+11v}{15}}$	8	in
5	$P_{-1,0}, P_{-\frac{5}{19},\frac{u+10v}{19}}, P_{-\frac{1}{19},\frac{2u+v}{19}}, P_{-\frac{20}{19},\frac{2u+v}{19}}$, $P_{-\frac{7}{19},\frac{3u+11v}{19}}, P_{-\frac{4}{19},\frac{4u+2v}{19}}, P_{-\frac{23}{19},\frac{4u+2v}{19}}, P_{-\frac{11}{19},\frac{5u+12v}{19}}$, $P_{-\frac{9}{19},\frac{6u+3v}{19}}, P_{-\frac{17}{19},\frac{7u+13v}{19}} P_{-\frac{16}{19},\frac{8u+4v}{19}}, P_{-\frac{6}{19},\frac{9u+14v}{19}}$	9	in