SIEGEL MODULAR FORMS OF DEGREE TWO AND LEVEL FIVE

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Abstract. We construct a ring of meromorphic Siegel modular forms of degree 2 and level 5, with singularities supported on an arrangement of Humbert surfaces, which is generated by four singular theta lifts of weights 1, 1, 2, 2 and their Jacobian. We use this to prove that the ring of holomorphic Siegel modular forms of degree 2 and level $\Gamma_0(5)$ is minimally generated by eighteen modular forms of weights $2, 4, 4, 4, 4, 4, 6, 6, 6, 6, 10, 11, 11, 11, 13, 13, 13, 15$.

1. Introduction

It is an interesting problem to determine the structure of rings of Siegel modular forms with respect to congruence subgroups. A famous theorem of Igusa [8] shows that every Siegel modular form of degree two and even weight for the full modular group $\mathrm{Sp}_4(\mathbb{Z})$ can be written uniquely as a polynomial in forms $\phi_4, \phi_6, \phi_{10}, \phi_{12}$ of weights $4, 6, 10, 12$, and that odd weight Siegel modular forms are precisely the products of even weight Siegel modular forms with a distinguished cusp form ψ_{35} of weight 35. It was proved by Aoki and Ibukiyama [1] that the rings of modular forms for the congruence subgroups

$$\Gamma_{0,1}^{(2)}(N) = \Big\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Sp}_4(\mathbb{Z}) : \; c \equiv 0\,(N), \; \det(a) \equiv \det(d) \equiv 1\,(N) \Big\}, \; N = 2, 3, 4$$

have an analogous structure: they are generated by four algebraic independent modular forms together with their Jacobian (or first Rankin–Cohen–Ibukiyama bracket). The rings $M_*(\Gamma_0^{(2)}(N))$ where

$$\Gamma_0^{(2)}(N) = \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Sp}_4(\mathbb{Z}) : \; c \equiv 0\,(N) \}$$

therefore have a simple structure as well.

The goal in this paper is to extend the methods of Aoki and Ibukiyama to level $N = 5$. This is not quite straightforward, as the natural underlying ring is no longer $M_*(\Gamma_{0,1}^{(2)}(5))$ but rather a ring $M_*^!(\Gamma_{0,1}^{(2)}(5))$ of meromorphic Siegel modular forms with singularities on Humbert surfaces. We will define a hyperplane arrangement $\mathcal{H}$ as the $\Gamma_0^{(2)}(5)$-orbit of the Humbert surface

$$\{ Z = \left(\begin{smallmatrix} \tau & z \\ z & w \end{smallmatrix}\right) \in \mathbb{H}_2 : \; \det(Z) = 1 - 5z \},$$

which, if one views points in $\mathbb{H}_2$ as parameterizing abelian surfaces, is a locus of principally polarized abelian surfaces with real multiplication that respects a $\Gamma_0^{(2)}(5)$ level structure. We then investigate the ring $M_*^!(\Gamma_{0,1}^{(2)}(5))$ of meromorphic Siegel modular forms on $\Gamma_{0,1}^{(2)}(5)$ with singularities supported on $\mathcal{H}$. Using a generalization of the modular Jacobian approach of [12], we prove in Theorem 3.6 that $M_*^!(\Gamma_{0,1}^{(2)}(5))$ is generated by four algebraically independent singular additive lifts f_1, f_2, g_1, g_2 of weights 1, 1, 2, 2 and by their Jacobian; in particular, the associated threefold $X_{0,1}^{(2)}(5)$ is rational. The local isomorphism from Sp_4 to $\mathrm{SO}(3, 2)$ and Borcherds' theory of orthogonal modular forms with singularities are essential. $\mathrm{Proj}(M_*^!(\Gamma_{0,1}^{(2)}(5)))$ is the Looijenga compactification [9] of the complement $(\mathbb{H}_2 \setminus \mathcal{H})/\Gamma_{0,1}^{(2)}(5)$, which plays a similar role to the Satake–Baily–Borel compactification of $Y_{0,1}^{(2)}(5)$.

It follows from the above that every Siegel modular form of level $\Gamma_0^{(2)}(5)$ can be expressed uniquely in terms of the basic forms f_1, f_2, g_1, g_2. It is not clear to the authors how to compute the ring $M_*(\Gamma_0^{(2)}(5))$ of (holomorphic) Siegel modular forms from this information alone; however, allowing a formula of Hashimoto

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[6] for the dimensions of cusp forms (itself an application of the Selberg trace formula), the ring structure becomes a straightforward Gröbner basis computation. We will prove that $M_*(\Gamma_0^{(2)}(5))$ is minimally generated by eighteen modular forms of weights $2, 4, 4, 4, 4, 4, 6, 6, 6, 6, 10, 11, 11, 11, 13, 13, 13, 15$ in Theorem 4.2.

This paper is organized as follows. In §2 we review the realization of Siegel modular groups as orthogonal groups and the theory of Borcherds lifts. In §3 we determine two rings of meromorphic Siegel modular forms. In §4 we use this to determine the ring of holomorphic Siegel modular forms for $\Gamma_0^{(2)}(5)$.

2. Theta lifts to Siegel modular forms of degree two

2.1. $\Gamma_0^{(2)}(N)$ **as an orthogonal group.** Recall that the Pfaffian of an antisymmetric (4×4)-matrix M is

$$\mathrm{pf}(M) = \mathrm{pf}\begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix} = af - be + cd.$$

We view pf as a quadratic form and define the associated bilinear form

$$\langle x, y\rangle = \mathrm{pf}(x+y) - \mathrm{pf}(x) - \mathrm{pf}(y).$$

The Pfaffian is invariant under conjugation $M \mapsto A^T M A$ by $A \in \mathrm{SL}_4(\mathbb{R})$ and this action identifies $\mathrm{SL}_4(\mathbb{R})$ with the Spin group $\mathrm{Spin}(\mathrm{pf})$. The symplectic group $\mathrm{Sp}_4(\mathbb{R})$, by definition, preserves

$$\mathcal{J} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

under conjugation, so it also preserves the orthogonal complement $\mathcal{J}^\perp$; and indeed it is exactly the Spin group of pf restricted to $\mathcal{J}^\perp$. If the entries of M are labelled as above then $M \in \mathcal{J}^\perp$ if and only if $b + e = 0$.

For any $N \in \mathbb{N}$, the group $\Gamma_0^{(2)}(N) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Sp}_4(\mathbb{Z}) :\ c \equiv 0\,(N)\}$ stabilizes the lattice

$$L = \left\{ M = \begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & -b \\ -b & -d & 0 & f \\ -c & b & -f & 0 \end{pmatrix} :\ a, b, c, d, f \in \mathbb{Z},\ a \equiv 0\,(N) \right\}$$

which is of type $U \oplus U(N) \oplus A_1$. By [7, §2] the special discriminant kernel $\widetilde{\mathrm{SO}}(L)$ of L is exactly the projective modular group $\Gamma_{0,1}^{(2)}(N)/\{\pm I\}$ under this identification, where

$$\widetilde{\mathrm{SO}}(L) = \{g \in \mathrm{SO}(L) :\ g(v) - v \in L \text{ for all } v \in L'\},$$
$$\Gamma_{0,1}^{(2)}(N) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0^{(2)}(N) :\ \det(a) \equiv 1\,(N)\}.$$

It follows that

$$\Gamma_0^{(2)}(N)/\{\pm I\} = \langle \widetilde{\mathrm{SO}}(L),\ \varepsilon_u :\ u \in (\mathbb{Z}/N\mathbb{Z})^\times\rangle$$

where ε_u is the matrix

$$\varepsilon_u = \begin{pmatrix} u & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ N & 0 & u^* & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \Gamma_0^{(2)}(N)$$

for any integer solutions u^*, b to $uu^* - Nb = 1$ (the choice does not matter). The map induced by ε_u on $L'/L \cong A_1'/A_1 \oplus U(N)'/U(N)$ acts trivially on A_1'/A_1 and acts on $U(N)'/U(N) \cong \mathbb{Z}/N\mathbb{Z} \oplus \mathbb{Z}/N\mathbb{Z}$ as the map

$$\varepsilon_u : \mathbb{Z}/N\mathbb{Z} \oplus \mathbb{Z}/N\mathbb{Z} \to \mathbb{Z}/N\mathbb{Z} \oplus \mathbb{Z}/N\mathbb{Z},\ (x, y) \mapsto (ux, u^{-1}y).$$

The symplectic group $\mathrm{Sp}_4(\mathbb{R})$ acts on the Siegel upper half-space $\mathbb{H}_2$ by Möbius transformations:

$$M \cdot Z = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot Z = (aZ + b)(cZ + d)^{-1}.$$

Let $j(M;Z) = \det(cZ+d)$ be the usual automorphy factor. We embed the Siegel upper half-space into $L \otimes \mathbb{C}$ as follows:

$$Z = \begin{pmatrix} \tau & z \\ z & w \end{pmatrix} \mapsto \mathcal{Z} := \phi(Z) := \begin{pmatrix} 0 & 1 & z & w \\ -1 & 0 & -\tau & -z \\ -z & \tau & 0 & \tau w - z^2 \\ -w & z & z^2 - \tau w & 0 \end{pmatrix}.$$

Then one has the relation

$$M^T \mathcal{Z} M = j(M;Z)\phi(M \cdot Z), \; M \in \mathrm{Sp}_4(\mathbb{R})$$

as one can check on any system of generators.

For any $\lambda \in L'$ of positive norm $D = Q(\lambda)$, the space

$$\{Z \in \mathbb{H}_2 : \; \mathcal{Z} \in \lambda^\perp\}$$

is known as a **Humbert surface** $H(D,\lambda)$ of discriminant D. If λ is written in the form $\left(\begin{smallmatrix} 0 & a & b & c \\ -a & 0 & d & -b \\ -b & -d & 0 & f \\ -c & b & -f & 0 \end{smallmatrix}\right)$ then

$$H(D,\lambda) = \{Z = \left(\begin{smallmatrix} \tau & z \\ z & w \end{smallmatrix}\right) \in \mathbb{H}_2 : \; a\det(Z) - c\tau + 2bz + dw + f = 0\}.$$

If γ instead is a coset of L'/L, then we define

$$H(D,\gamma) = \sum_{\substack{\lambda \in \gamma \\ \lambda \text{ primitive in } L' \\ Q(\lambda) = D}} H(D,\lambda).$$

These unions are locally finite and therefore descend to well-defined divisors on $\widetilde{\mathrm{O}}(L)\backslash\mathbb{H}_2$. We will use the notation $H(D,\pm\gamma)$ because $\lambda^\perp = (-\lambda)^\perp$ implies $H(D,\gamma) = H(D,-\gamma)$. Note that many references omit the condition that λ be primitive in L', so $H(D,\pm\gamma)$ satisfy inclusions; our divisors $H(D,\pm\gamma)$ do not.

2.2. **Theta lifts.** Let L be the lattice in the space of (4×4) antisymmetric matrices from the previous subsection. The **weight k theta kernel** is

$$\Theta_k(\tau;Z) = \frac{\pi^k}{\det(V)^k \Gamma(k)} \sum_{\lambda \in L'} \langle \lambda, \mathcal{Z}\rangle^k e^{-\frac{\pi y}{\det(V)}|\langle \lambda, \mathcal{Z}\rangle|^2} e^{2\pi i \overline{\tau} \mathrm{pf}(\lambda)} \mathfrak{e}_\lambda,$$

where $\tau = x + iy \in \mathbb{H}$; and $Z = U + iV \in \mathbb{H}_2$; and $\mathcal{Z}$ is the image of Z in $L \otimes \mathbb{C}$. By applying a theorem of Vignéras on indefinite theta series [11] one sees that Θ_k transforms like a modular form of weight $\kappa := k - 1/2$ with respect to the Weil representation ρ_L. On the other hand, for any $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0^{(2)}(N)$,

$$\begin{aligned} \Theta_k(\tau; M\cdot Z) &= \frac{\pi^k}{\det \mathrm{im}(M\cdot Z)^k \Gamma(k)} \sum_{\lambda \in L'} \det(cZ+d)^{-k} \langle \lambda, M^T \mathcal{Z} M\rangle e^{-\frac{\pi y}{\det(V)}|\langle \lambda, M^T \mathcal{Z} M\rangle|^2} e^{2\pi i \overline{\tau}\mathrm{pf}(\lambda)} \mathfrak{e}_\lambda \\ &= \frac{\pi^k}{\det(V)^k \Gamma(k)} \overline{\det(cZ+d)^k} \sum_{\lambda \in L'} \langle M^{-T}\lambda M^{-1}, \mathcal{Z}\rangle^k e^{-\frac{\pi y}{\det(V)}|\langle M^{-T}\lambda M^{-1}, \mathcal{Z}\rangle|^2} e^{2\pi i \overline{\tau}\mathrm{pf}(M^{-T}\lambda M^{-1})} \mathfrak{e}_\lambda \\ &= \overline{\det(cZ+d)^k} \sigma(M) \Theta_k(\tau;Z), \end{aligned}$$

where σ is the map

$$\sigma : \Gamma_0^{(2)}(N) \longrightarrow \mathrm{Aut}\,\mathbb{C}[L'/L], \;\; \sigma(M)\mathfrak{e}_\lambda := \mathfrak{e}_{M^T \lambda M}.$$

Following Borcherds [3] one defines the theta lift of a vector-valued modular form F with a pole at ∞ as the regularized integral of F against the kernel Θ_k:

Definition 2.1. (i) The *Weil representation* ρ_L associated to an even lattice (L,Q) is the representation $\rho : \mathrm{Mp}_2(\mathbb{Z}) \to \mathrm{GL}\,\mathbb{C}[L'/L]$ defined by

$$\rho\left(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), 1\right) \mathfrak{e}_\gamma = e^{-2\pi i Q(\gamma)} \mathfrak{e}_\gamma;$$

$$\rho\left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), \sqrt{\tau}\right) \mathfrak{e}_\gamma = e^{\pi i \mathrm{sig}(L)/4} |L'/L|^{-1/2} \sum_{\beta \in L'/L} e^{2\pi i \langle \gamma, \beta\rangle} \mathfrak{e}_\beta.$$

Here $\mathrm{Mp}_2(\mathbb{Z})$ is the metaplectic group of pairs (M, ϕ) where $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and ϕ is a square root of $c\tau + d$, and $\mathfrak{e}_\gamma$, $\gamma \in L'/L$ is the standard basis of the group ring $\mathbb{C}[L'/L]$.
(ii) A *nearly-holomorphic vector-valued modular form* for ρ_L is a holomorphic function $F : \mathbb{H} \to \mathbb{C}[L'/L]$ which satisfies

$$F((M, \phi) \cdot \tau) = \phi(\tau)^{2k} \rho_L(M) F(\tau), \quad (M, \phi) \in \mathrm{Mp}_2(\mathbb{Z})$$

and which is meromorphic at the cusp ∞, i.e. its Fourier series has only finitely many negative exponents.
(iii) Let $k \geq 1$ and let $F \in M^!_\kappa(\rho_L)$ be a nearly-holomorphic modular form. The **(singular) theta lift** of F is

$$\Phi_F(Z) = \int^{\mathrm{reg}}_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} \langle F(\tau), \Theta_k(\tau; Z)\rangle y^\kappa \frac{\mathrm{d}x\, \mathrm{d}y}{y^2}.$$

Here the regularization means one takes the limit as $w \to \infty$ of the integral over $\mathcal{F}_w = \{\tau = x + iy \in \mathbb{H} : x^2 + y^2 \geq 1,\ |x| \leq 1/2,\ y \leq w\}$; in effect, it means one integrates first with respect to x, which mollifies the contribution of the principal part of F to the integral; and then secondly with respect to y. The behavior of the theta lift under Möbius transformations is

$$\begin{aligned} \Phi_F(M \cdot Z) &= \int^{\mathrm{reg}} \langle F(\tau), \Theta_k(\tau; M \cdot Z)\rangle\ y^{\kappa-2}\, \mathrm{d}x\, \mathrm{d}y \\ &= \det(cZ + d)^k \int^{\mathrm{reg}} \sum_{\gamma \in L'/L} F_\gamma(\tau) \overline{\Theta_{k; M^{-T}\gamma M^{-1}}(\tau; Z)}\, y^{\kappa-2}\, \mathrm{d}x\, \mathrm{d}y \\ &= \det(cZ + d)^k \Phi_{\sigma(M)^{-1}F}(Z). \end{aligned}$$

Therefore, the singular theta lift Φ_F defines a meromorphic Siegel modular form of weight k on the subgroup of $\Gamma_0^{(2)}(N)$ that fixes F, with singularities of multiplicity k along Humbert surfaces associated to the principal part of the input F. This is the so-called Borcherds additive lift. We refer to [3, Theorem 14.3] for more details. Since the Borcherds additive lift is a generalization of the Gritsenko lift [5], we also call it the singular Gritsenko lift. When the input F has weight $\kappa = -\frac{1}{2}$ (i.e. $k = 0$), the modified exponential of Φ_F defines a remarkable modular form which has an infinite product expansion (cf. [3, Theorem 13.3]), called a *Borcherds product*, or more specifically the Borcherds lift of F. In this paper we will need both types of singular theta lifts.

Remark 2.2. It is often useful to consider the *pullback* or restriction of a Siegel modular form to a Humbert surface, the result being traditionally interpreted as a Hilbert modular form attached to a real-quadratic field. From the point of view of orthogonal modular forms this is very simple: to restrict a form $\Phi(Z)$ to the sublattice $v^\perp$ (with $v \in L'$) one simply restricts to Z satisfying $\langle Z, v\rangle = 0$.

The pullback of a theta lift Φ_F as above is again a theta lift, $\Phi_{\vartheta F}$, where $\vartheta F \in M_{\kappa+1/2}(\rho_{v^\perp})$ is the *theta contraction*, obtained roughly by tensoring F with a unary theta series and averaging out. The important point is that one can check rigorously whether a theta lift Φ_F vanishes identically on a Heegner divisor, with the computations taking place only on the level of vector-valued modular forms.

3. The ring of meromorphic Siegel modular forms of level 5

We consider the ring $M^!_*(\Gamma_0^{(2)}(5))$ of meromorphic Siegel modular forms of level $\Gamma_0^{(2)}(5)$ whose poles may lie only on the orbit $\mathcal{H}$ of the Humbert surface

$$\{Z = \left(\begin{smallmatrix} \tau & z \\ z & w \end{smallmatrix}\right) \in \mathbb{H}_2 :\ \det(Z) = 1 - 5z\},$$

which is a locus of principally polarized RM abelian surfaces with $\Gamma_0^{(2)}(5)$ level structure. In view of our discussion earlier, $\mathcal{H}$ splits as the union of two irreducible $\Gamma_{0,1}^{(2)}(5)$-orbits of Humbert surfaces:

$$\mathcal{H} = H(1/20, \pm\gamma_1) + H(1/20, \pm\gamma_2),$$

each invariant under the discriminant kernel of $L = U \oplus U(5) \oplus A_1$, where we have fixed any coset $\gamma_1 \in L'/L$ of norm $1/20 + \mathbb{Z}$ and define $\gamma_2 = \varepsilon_2(\gamma_1)$. The Humbert surface $H_{1/5}$ of discriminant $1/5$, the orbit of $\{\left(\begin{smallmatrix} \tau & z \\ z & w \end{smallmatrix}\right) \in \mathbb{H}_2 :\ \tau = 2z\}$ under $\Gamma_0^{(2)}(5)$, also splits into two $\Gamma_{0,1}^{(2)}(5)$-invariant divisors:

$$H_{1/5} = H(1/5, \pm\delta_1) + H(1/5, \pm\delta_2),$$

where $\delta_n = 2\gamma_n \in L'/L$.

For a finite-index subgroup $\Gamma \leq \Gamma_0^{(2)}(5)$ or $\Gamma \leq \mathrm{O}(L)$, we define $M^!_*(\Gamma, \chi)$ to be the ring of meromorphic forms, holomorphic away from $\mathcal{H}$, which are modular under Γ with character χ.

We first prove a form of Koecher's principle for meromorphic modular forms with poles supported on $\mathcal{H}$.

Lemma 3.1. *Let $f \in M^!_k(\Gamma^{(2)}_{0,1}(5), \chi)$. If k is negative, then f is identically zero. If $k = 0$, then f is constant.*

Proof. We prove the lemma in the context of $\mathrm{O}(3, 2)$. Let v and $u \neq \pm v$ be primitive vectors of norm $1/20$ in L', such that $v^\perp, u^\perp \in \mathcal{H}$. Suppose that f is not identically zero and has poles of multiplicity c_v along $v^\perp$. We denote the intersection of $v^\perp$ and the symmetric domain $\mathbb{H}_2$ (resp. the lattice L) by $v^\perp \cap \mathbb{H}_2$ (resp. L_v). Then L_v is a lattice of signature $(2, 2)$ and discriminant 5, equivalent to the lattice $U + \mathbb{Z}[(1 + \sqrt{5})/2]$ where the quadratic form is the field norm. It is easy to see that the space $L_v \otimes \mathbb{Q}$ contains no isotropic planes, so the Koecher principle holds for modular forms on $\widetilde{\mathrm{O}}(L_v)$. We find that the projection of u in L_v has non-positive norm, which implies that the intersection of $u^\perp$ and $v^\perp \cap \mathbb{H}_2 \cong \mathbb{H} \times \mathbb{H}$ is empty. Thus the quasi-pullback of f to $v^\perp \cap \mathbb{H}_2$, i.e. the leading term in the power series expansion about that hyperplane, is a nonzero holomorphic modular form of weight $k - c_v$. By Koecher's principle we conclude $k - c_v \geq 0$ and therefore $k \geq 0$, and when $k = 0$, we must have $c_v = 0$ and thus f is holomorphic and must be constant (by Koecher's principle on $\widetilde{\mathrm{O}}(L)$). $\square$

We now construct some basic modular forms using Borcherds additive lifts (singular Gritsenko lifts) and Borcherds products.

Lemma 3.2. *There are singular Gritsenko lifts f_1, f_2 of weight one on $\widetilde{\mathrm{O}}(L)$ whose divisors are exactly*

$$\mathrm{div}(f_1) = -H(1/20, \pm\gamma_1) + 4H(1/20, \pm\gamma_2) + H(1/5, \pm\delta_1)$$

and

$$\mathrm{div}(f_2) = 4H(1/20, \pm\gamma_1) - H(1/20, \pm\gamma_2) + H(1/5, \pm\delta_2).$$

Proof. Using the algorithm of [13] we find a nearly-holomorphic modular form of weight 1/2 for the Weil representation associated to L whose Fourier expansion takes the form

$$2q^{-1/20}(\mathfrak{e}_{\gamma_1} - \mathfrak{e}_{-\gamma_1}) + O(q^{1/20}),$$

which is mapped under the Gritsenko lift to a meromorphic form f_1 with simple poles only on $H(1/20, \pm\gamma_1)$ and $H(1/20, \pm\gamma_4)$. Applying the automorphism ε_2 on L'/L to the input into f_1 yields the input into f_2.

On the other hand, we found a nearly-holomorphic modular form of weight $-1/2$ whose principal part at ∞ is

$$2\mathfrak{e}_0 - 2q^{-1/20}(\mathfrak{e}_{\gamma_1} + \mathfrak{e}_{-\gamma_1}) + 4q^{-1/20}(\mathfrak{e}_{\gamma_2} + \mathfrak{e}_{-\gamma_2}) + q^{-1/5}(\mathfrak{e}_{\delta_1} + \mathfrak{e}_{-\delta_1}),$$

which is mapped under the Borcherds lift to a meromorphic modular form F_1 (possibly with character) of weight one and the claimed divisor. By taking theta contractions of the input form one finds that f_1 vanishes on $H(1/5, \pm\delta_1)$. Then the quotient f_1/F_1 lies in $M^!_0(\widetilde{\mathrm{O}}(L), \chi)$ so it is constant by Lemma 3.1. $\square$

Remark 3.3. The Fourier expansions of f_1 and f_2 begin

$$f_1\left(\left(\begin{smallmatrix} \tau & z \\ z & w \end{smallmatrix}\right)\right) = 1 + 3q + 3s + 4q^2 + (2r^{-1} + 6 + 2r)qs + 4s^2 + O(q, s)^3;$$
$$f_2\left(\left(\begin{smallmatrix} \tau & z \\ z & w \end{smallmatrix}\right)\right) = q - s - 2q^2 + 2s^2 + 4q^3 + (4r^2 + 2 + 4r)qs(q - s) - 4s^3 + O(q, s)^4;$$

where as usual $q = e^{2\pi i\tau}$, $r = e^{2\pi iz}$, $s = e^{2\pi iw}$. For more coefficients see Figure 1 below. Setting $s = 0$ one obtains the (holomorphic) modular forms

$$\Phi(f_1) = 1 + 3q + 4q^2 \pm ..., \; \Phi(f_2) = q - 2q^2 + 4q^3 \pm ...$$

of weight one and level $\Gamma_1(5)$ which freely generate the ring $M_*(\Gamma_1(5))$.

There are nine Heegner divisors of discriminant 1/4. One is the mirror of the reflective vector $r = 1/2 \in A_1'$, represented by the diagonal in $\mathbb{H}_2$, and the other eight are of the form $H(1/4, r + \gamma)$ where γ are the isotropic cosets of $U(5)'/U(5)$. It will be convenient to fix concrete representatives. We take the Gram matrix $\mathbf{S} = \left(\begin{smallmatrix} 0 & 0 & 5 \\ 0 & 2 & 0 \\ 5 & 0 & 0 \end{smallmatrix}\right)$ for $U(5) \oplus A_1$, such that $L'/L \cong \mathbf{S}^{-1}\mathbb{Z}^3/\mathbb{Z}^3$ and fix the cosets

$$\gamma_1 = (1/5, 1/2, 4/5) + L \qquad \gamma_2 = (2/5, 1/2, 2/5) + L$$
$$\gamma_3 = (3/5, 1/2, 3/5) + L \qquad \gamma_4 = (4/5, 1/2, 1/5) + L$$

of norm $1/20 + \mathbb{Z}$. The norm $1/4$ cosets other than r are labelled

$$\alpha_n = (n/5, 1/2, 0) + L,\ \beta_n = (0, 1/2, n/5) + L,\ n \in \{1, 2, 3, 4\}.$$

Lemma 3.4. *There are singular Gritsenko lifts g_1, g_2, h_1, h_2 of weight two on $\widetilde{\mathrm{O}}(L)$ whose divisors are exactly*

$$\begin{aligned}
\operatorname{div} g_1 &= 3H(1/20, \pm\gamma_1) - 2H(1/20, \pm\gamma_2) + H(1/4, \pm\alpha_2);\\
\operatorname{div} g_2 &= -2H(1/20, \pm\gamma_1) + 3H(1/20, \pm\gamma_2) + H(1/4, \pm\alpha_1);\\
\operatorname{div} h_1 &= 3H(1/20, \pm\gamma_1) - 2H(1/20, \pm\gamma_2) + H(1/4, \pm\beta_2);\\
\operatorname{div} h_2 &= -2H(1/20, \pm\gamma_1) + 3H(1/20, \pm\gamma_2) + H(1/4, \pm\beta_1).
\end{aligned}$$

Proof. The proof is essentially the same argument as Lemma 3.2. Using the pullback trick, one constructs weight two Gritsenko lifts which vanish on the claimed discriminant $1/4$ Heegner divisors. Then one constructs Borcherds products of weight two with the claimed divisors. The respective quotients lie in $M^!_0(\widetilde{\mathrm{O}}(L), \chi)$ and are therefore constant by Lemma 3.1. To determine the precise (nearly-holomorphic) vector-valued modular forms which lift to g_1, g_2, h_1, h_2, one only needs to compute the four-dimensional space of nearly-holomorphic forms of weight $3/2$ for ρ_L with a pole of order at most $1/20$ at ∞, and identify the unique (up to scalar) forms whose pullback to $\alpha_n^\perp$ or $\beta_n^\perp$ is respectively zero. The input forms G_1, G_2, H_1, H_2 can be chosen such that their Fourier expansions begin as follows:

$$\begin{aligned}
G_1 :&\quad q^{-1/20}(\mathfrak{e}_{\gamma_2} + \mathfrak{e}_{\gamma_3}) + (\mathfrak{e}_{(0,0,1/5)} + \mathfrak{e}_{(0,0,4/5)} - \mathfrak{e}_{(0,0,2/5)} + \mathfrak{e}_{(0,0,3/5)}) + O(q^{1/20})\\
G_2 :&\quad q^{-1/20}(\mathfrak{e}_{\gamma_1} + \mathfrak{e}_{\gamma_4}) + (\mathfrak{e}_{(0,0,2/5)} + \mathfrak{e}_{(0,0,3/5)} - \mathfrak{e}_{(0,0,1/5)} + \mathfrak{e}_{(0,0,4/5)}) + O(q^{1/20})\\
H_1 :&\quad q^{-1/20}(\mathfrak{e}_{\gamma_2} + \mathfrak{e}_{\gamma_3}) + (\mathfrak{e}_{(1/5,0,0)} + \mathfrak{e}_{(4/5,0,0)} - \mathfrak{e}_{(2/5,0,0)} + \mathfrak{e}_{(3/5,0,0)}) + O(q^{1/20})\\
H_2 :&\quad q^{-1/20}(\mathfrak{e}_{\gamma_1} + \mathfrak{e}_{\gamma_4}) + (\mathfrak{e}_{(2/5,0,0)} + \mathfrak{e}_{(3/5,0,0)} - \mathfrak{e}_{(1/5,0,0)} + \mathfrak{e}_{(4/5,0,0)}) + O(q^{1/20}).
\end{aligned}$$

These expansions determine G_1, G_2, H_1, H_2 uniquely because there are no vector-valued cusp forms of weight $3/2$ for ρ_L. ∎

Remark 3.5. The Fourier expansions of g_1, g_2, h_1, h_2 begin as follows:

$$\begin{aligned}
g_1\left(\left(\begin{smallmatrix} \tau & z \\ z & w \end{smallmatrix}\right)\right) &= q + q^2 - 5qs + 2q^3 + (-3r^{-1} + 1 - 3r)q^2 s + (-r^{-1} + 7 - r)qs^2 + O(q, s)^4;\\
g_2\left(\left(\begin{smallmatrix} \tau & z \\ z & w \end{smallmatrix}\right)\right) &= -q - q^2 + (r^{-1} + 3 + r)qs - 2q^3 + (-r^{-1} + 7 - r)q^2 s + (-3r^{-1} + 1 - 3r)qs^2 + O(q, s)^4;\\
h_1\left(\left(\begin{smallmatrix} \tau & z \\ z & w \end{smallmatrix}\right)\right) &= s - 5qs + s^2 + (-r^{-1} + 7 - r)q^2 s + (-3r^{-1} + 1 - 3r)qs^2 + 2s^3 + O(q, s)^4;\\
h_2\left(\left(\begin{smallmatrix} \tau & z \\ z & w \end{smallmatrix}\right)\right) &= -s + (r^{-1} + 3 + r)qs - s^3 + (-3r^{-1} + 1 - 3r)q^2 s + (-r^{-1} + 7 - r)qs^2 - 2s^3 + O(q, s)^4.
\end{aligned}$$

We can now determine the structure of $M^!_*(\Gamma^{(2)}_{0,1}(5))$. Recall that $\Gamma^{(2)}_{0,1}(5)/\{\pm I\} \cong \widetilde{\mathrm{SO}}(L)$. The decomposition

$$M^!_k(\widetilde{\mathrm{SO}}(L)) = M^!_k(\widetilde{\mathrm{O}}(L)) \oplus M^!_k(\widetilde{\mathrm{O}}(L), \det),$$

suggests that we first consider the ring of modular forms for the discriminant kernel $\widetilde{\mathrm{O}}(L)$. We will show that $M^!_*(\widetilde{\mathrm{O}}(L))$ is freely generated using a generalization of the modular Jacobian approach of [12, Theorem 5.1]. We briefly introduce the main objects of this approach. For any four $\psi_i \in M^!_{k_i}(\widetilde{\mathrm{O}}(L))$ with $1 \leq i \leq 4$, their Jacobian (see [12, Theorem 2.5] and [1, Proposition 2.1])

$$J(\psi_1, \psi_2, \psi_3, \psi_4) = \begin{vmatrix} k_1\psi_1 & k_2\psi_2 & k_3\psi_3 & k_4\psi_4 \\ \frac{\partial \psi_1}{\partial \tau} & \frac{\partial \psi_2}{\partial \tau} & \frac{\partial \psi_3}{\partial \tau} & \frac{\partial \psi_4}{\partial \tau} \\ \frac{\partial \psi_1}{\partial z} & \frac{\partial \psi_2}{\partial z} & \frac{\partial \psi_3}{\partial z} & \frac{\partial \psi_4}{\partial z} \\ \frac{\partial \psi_1}{\partial w} & \frac{\partial \psi_2}{\partial w} & \frac{\partial \psi_3}{\partial w} & \frac{\partial \psi_4}{\partial w} \end{vmatrix}$$

lies in $M^!_{k_1+k_2+k_3+k_4+3}(\widetilde{\mathrm{O}}(L), \det)$. The Jacobian $J(\psi_1, \psi_2, \psi_3, \psi_4)$ is not identically zero if and only if the four forms ψ_i are algebraically independent over $\mathbb{C}$.

The discriminant kernel $\widetilde{\mathrm{O}}(L)$ contains reflections associated to vectors of norm 1 in L (the so-called 2-reflections)

$$\sigma_v : x \mapsto x - (x, v)v.$$

The hyperplane $v^\perp$ is called the mirror of the reflection σ_v. Since $\det(\sigma_v) = -1$, the chain rule implies that the above Jacobian vanishes on all mirrors of 2-reflections. Conversely, the main theorem of [12], and its generalization to meromorphic modular forms with constrained poles, implies that

Theorem 3.6. *The ring $M^!_*(\widetilde{\mathrm{O}}(L))$ is a free algebra:*

$$M^!_*(\widetilde{\mathrm{O}}(L)) = \mathbb{C}[f_1, f_2, g_1, g_2].$$

Define $J := J(f_1, f_2, g_1, g_2)$. Then

$$M^!_*(\Gamma^{(2)}_{0,1}(5)) = \mathbb{C}[f_1, f_2, g_1, g_2, J].$$

Proof. The Jacobian J of f_1, f_2, g_1, g_2 has weight 9 and vanishes on the mirrors of 2-reflections, which form a union of Heegner divisors of discriminants $1/4$ and 1 denoted Δ. Using the Fourier expansions of the forms it is easy to check that J is not identically zero. Using the algorithm of [13] we find a Borcherds product J_0 with divisor

$$\mathrm{div}\, J_0 = \Delta + 6H(1/20, \pm\gamma_1) + 6H(1/20, \pm\gamma_2).$$

The quotient J/J_0 lies in $M^!_0(\Gamma^{(2)}_{0,1}(5), \chi)$ and is therefore a constant denoted c by Lemma 3.1. We will now prove the claim by an argument which appeared essentially in [12]. Suppose that $M^!_*(\widetilde{\mathrm{O}}(L))$ was not generated by $h_1 := f_1$, $h_2 := f_2$, $h_3 := g_1$ and $h_4 := g_2$, and let $h_5 \in M^!_{k_5}(\widetilde{\mathrm{O}}(L))$ be a modular form of minimal weight which is not contained in $\mathbb{C}[f_1, f_2, g_1, g_2]$. Set $k_1 = k_2 = 1$ and $k_3 = k_4 = 2$, such that k_i is the weight of h_i. For $1 \le j \le 5$ we define J_j as the Jacobian of the four modular forms h_i omitting h_j, such that $cJ_0 = J = J_5$. It is clear that $g_j := J_j/J$ is a modular form on $\widetilde{\mathrm{O}}(L)$ with poles supported on $\mathcal{H}$. We compute the determinant and find the identity

$$0 = \det \begin{pmatrix} k_1 h_1 & ... & k_4 h_4 & k_5 h_5 \\ k_1 h_1 & ... & k_4 h_4 & k_5 h_5 \\ \nabla h_1 & ... & \nabla h_4 & \nabla h_5 \end{pmatrix} = \sum_{i=1}^5 (-1)^{i+1} k_i h_i J_i.$$

Since $J_i = J g_i$ and $g_5 = 1$, we have

$$\sum_{i=1}^5 (-1)^{i+1} k_t h_t g_t = 0, \quad \text{i.e} \quad k_5 h_5 = \sum_{i=1}^4 (-1)^i h_i g_i.$$

Since h_5 was chosen to have minimal weight, $g_i \in \mathbb{C}[h_1, h_2, h_3, h_4]$ for all i, and thus $h_5 \in \mathbb{C}[h_1, h_2, h_3, h_4]$, which is a contradiction.

Now any $h \in M^!_k(\widetilde{\mathrm{O}}(L), \det)$ vanishes on all mirrors of 2-reflections, which implies that $h/J \in M^!_{k-9}(\widetilde{\mathrm{O}}(L))$. Therefore

$$M^!_*(\Gamma^{(2)}_{0,1}(5)) = M^!_*(\widetilde{\mathrm{SO}}(L)) = M^!_*(\widetilde{\mathrm{O}}(L)) \oplus M^!_*(\widetilde{\mathrm{O}}(L), \det)$$

is generated by f_1, f_2, g_1, g_2 and J. ☐

Remark 3.7. The weight two singular Gritsenko lifts satisfy the relations

$$g_1 - h_1 = h_2 - g_2 = f_1 f_2.$$

The product $f_1 f_2$ is holomorphic and in fact itself a Gritsenko lift; but it has a quadratic character under $\Gamma^{(2)}_0(5)$. There is a unique normalized Siegel modular form e_2 of weight two for $\Gamma^{(2)}_0(5)$, which can be constructed as the Gritsenko lift of the unique vector-valued modular form of weight $3/2$ for ρ_L invariant under all automorphisms of the discriminant form. (The uniqueness follows from Corollary 3.8 blow.) In terms of the generators of $M^!_*(\Gamma^{(2)}_{0,1}(5))$, a computation shows

$$e_2 = f_1^2 + f_2^2 - 4(g_1 + g_2).$$

Corollary 3.8. *The ring $M_*^!(\Gamma_0^{(2)}(5))$ is minimally generated in weights* $2, 2, 4, 4, 4, 4, 4, 11, 11, 11$ *by the ten forms*

$$\begin{array}{lllll} f_1^2 + f_2^2, & e_2, & f_1^2 g_1 + f_2^2 g_2, & f_1 f_2(g_1 - g_2), & f_1 f_2(f_1 - f_2)(f_1 + f_2), \\ f_1^2 f_2^2, & g_1 g_2, & J f_1 f_2, & J(f_1^2 - f_2^2), & J(g_1 - g_2). \end{array}$$

Proof. The group $\Gamma_0^{(2)}(5)$ is generated by the special discriminant kernel of L and by the order four automorphism ε_2 which acts on the generators of $M_*^!(\Gamma_{0,1}^{(2)}(5))$ by

$$\varepsilon_2 : f_1 \mapsto f_2,\ f_2 \mapsto -f_1,\ g_1 \mapsto g_2,\ g_2 \mapsto g_1,\ J \mapsto -J$$

as one can see on the input functions into the Gritsenko lifts. We conclude the action of ε_2 on J (as a Jacobian) from the actions of ε_2 on f_1, f_2, g_1 and g_2. The expressions in f_1, f_2, g_1, g_2, J in the claim generate the ring of invariants under this action. □

Remark 3.9. The same argument shows that the kernel of $\varepsilon_2^2 = \varepsilon_4$ is generated by J and by the weight two forms

$$f_1^2, f_2^2, f_1 f_2, g_1, g_2.$$

This corresponds to the quadratic Nebentypus $\chi\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) = \left(\frac{5}{\det d}\right)$ on $\Gamma_0^{(2)}(5)$. Note that $f_1 f_2$ is the Siegel Eisenstein series of weight two for the character χ, and that the Jacobian J is the unique cusp form of weight nine for χ up to scalars.

Remark 3.10. There is a seven-dimensional space of modular forms of weight $7/2$ for ρ_L, and a four-dimensional subspace on which ε_2 acts trivially, so the weight four Maass space for $\Gamma_0^{(2)}(5)$ is four-dimensional. Using the structure theorem above we can identify it by comparing only a few Fourier coefficients:

$$\mathrm{Maass}_4 = \mathrm{Span}(\phi_1, \phi_2, \phi_3, \phi_4)$$

where

$$\begin{aligned} \phi_1 &= e_2^2 + f_1^2 f_2^2; \\ \phi_2 &= f_1^2 g_1 + f_2^2 g_2 - 2 g_1 g_2; \\ \phi_3 &= f_1 f_2(f_1^2 - 2 f_1 f_2 - f_2^2 + 2 g_1 - 2 g_2); \\ \phi_4 &= 2 g_1 g_2 + f_1 f_2(g_1 - g_2). \end{aligned}$$

The form ϕ_4 is a cusp form and indeed spans $S_4(\Gamma_0^{(2)}(5))$, which was shown to be one-dimensional by Poor and Yuen [10].

4. The ring of holomorphic Siegel modular forms of level 5

In this section we investigate the ring $M_*(\Gamma_0^{(2)}(5))$ of holomorphic Siegel modular forms for $\Gamma_0^{(2)}(5)$. We will need the Hilbert–Poincaré series for this ring, which can be derived from dimension formulas available in the literature.

Theorem 4.1. *The Hilbert–Poincaré series of dimensions of modular forms for $\Gamma_0^{(2)}(5)$ is*

$$\sum_{k=0}^{\infty} \dim M_k(\Gamma_0^{(2)}(5)) t^k = \frac{(1-t)^2(1+t^7)P(t)}{(1-t^2)^2(1-t^3)(1-t^4)^2(1-t^5)}$$

where $P(t)$ is the irreducible palindromic polynomial

$$P(t) = 1 + 2t + 2t^2 + t^3 + 3t^4 + 5t^5 + 8t^6 + 8t^7 + 8t^8 + 5t^9 + 3t^{10} + t^{11} + 2t^{12} + 2t^{13} + t^{14}.$$

The first values of $\dim M_k(\Gamma_0^{(2)}(5))$ are given in Table 1 below.

Proof. The dimensions of the spaces of cusp forms of weight $k \geq 5$ have been computed in closed form by Hashimoto by means of the Selberg trace formula and in lower weights by Poor and Yuen [10]: we have

Table 1. $\dim M_k(\Gamma_0^{(2)}(5))$

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
dim	0	1	0	6	0	10	0	22	0	34	3	57	6	79	16	117	25	153	45

$\dim S_4(\Gamma_0^{(2)}(5)) = 1$ and $\dim S_k(\Gamma_0^{(2)}(5)) = 0$ for $k \leq 3$. All odd-weight modular forms are cusp forms, and by a more general theorem of Böcherer–Ibukiyama [2], for even $k > 2$,

$$\dim M_k(\Gamma_0^{(2)}(5)) = \dim S_k(\Gamma_0^{(2)}(5)) + 2 \cdot \dim S_k(\Gamma_0(5)) + 3.$$

□

We can now determine the generators of $M_*(\Gamma_0^{(2)}(5))$ using Corollary 3.8 together with the above generating series.

Theorem 4.2. *The ring of Siegel modular forms of level $\Gamma_0^{(2)}(5)$ is minimally generated by the weight two form*

$$e_2 = f_1^2 + f_2^2 - 4g_1 - 4g_2,$$

five weight four forms

$$f_1^2 g_1 + f_2^2 g_2, \quad f_1 f_2(g_1 - g_2), \quad f_1 f_2(f_1^2 - f_2^2), \quad f_1^2 f_2^2, \quad g_1 g_2,$$

four weight six forms

$$f_1^2 f_2^2 (g_1 + g_2), \quad f_1^3 f_2 g_1 - f_2^3 f_1 g_2, \quad f_1^2 g_1^2 + f_2^2 g_2^2, \quad g_1 g_2 (f_1^2 + f_2^2),$$

the weight ten form

$$f_1^2 f_2^2 (f_1^2 + f_2^2)^3,$$

three weight eleven forms

$$f_1 f_2 J, \quad (f_1^2 - f_2^2) J, \quad (g_1 - g_2) J,$$

three weight thirteen forms

$$(f_1^2 + f_2^2) f_1 f_2 J, \quad (f_1^4 - f_2^4) J, \quad (f_1^2 + f_2^2)(g_1 - g_2) J,$$

and the weight fifteen form

$$(f_1^2 - f_2^2)^3 J.$$

Proof. From the divisors of f_1, f_2, g_1, g_2 and J, it is easy to see that all of the forms above (except for e_2, which was discussed in the previous section) are holomorphic and ε_2-invariant. The Hilbert series of this ring was computed in Macaulay2 [4] and coincides exactly with the series predicted by Theorem 4.1, so we can conclude that these forms are sufficient to generate all holomorphic Siegel modular forms. □

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5. Appendix

On the following pages, we list more Fourier coefficients of the basic meromorphic forms f_1, f_2, g_1, g_2, as well as the unique expression for J^2 as a polynomial in these forms.

Note that the polynomial representing J^2 must split into two irreducible factors, corresponding to the two classes of reflections whose mirrors lie in the divisor of J. One of these factors is $g_1 + g_2$.

a	b	c	f_1	f_2	g_1	g_2
0	0	0	1	0	0	0
1	0	0	3	1	1	−1
0	0	1	3	−1	0	0
2	0	0	4	−2	1	−1
1	−1	1	2	0	0	1
1	0	1	6	0	−5	3
1	1	1	2	0	0	1
0	0	2	4	2	0	0
3	0	0	2	4	2	−2
2	−1	1	4	−4	−3	−1
2	0	1	2	−2	1	7
2	1	1	4	−4	−3	−1
1	−1	2	4	4	−1	−3
1	0	2	2	2	7	1
1	1	2	4	4	−1	−3
0	0	3	2	−4	0	0
4	0	0	1	−3	3	−3
3	−1	1	6	6	−2	4
3	0	1	−2	−2	−6	2
3	1	1	6	6	−2	4
2	−2	2	2	0	0	2
2	−1	2	8	0	10	−2
2	0	2	0	0	−15	−5
2	1	2	8	0	10	−2
2	2	2	2	0	0	2
1	−1	3	6	−6	4	−2
1	0	3	−2	2	2	−6
1	1	3	6	−6	4	−2
0	0	4	1	3	0	0
5	0	0	3	1	5	−5
4	−1	1	0	−8	−2	10
4	0	1	0	6	−11	−5
4	1	1	0	−8	−2	10
3	−2	2	0	6	3	−5
3	−1	2	0	0	−10	10
3	0	2	0	8	24	−20
3	1	2	0	0	−10	10
3	2	2	0	6	3	−5
2	−2	3	0	−6	3	−5
2	−1	3	0	0	−10	10
2	0	3	0	−8	24	−20
2	1	3	0	0	−10	10
2	2	3	0	−6	3	−5
1	−1	4	0	8	−2	10
1	0	4	0	−6	−11	−5
1	1	4	0	8	−2	10
0	0	5	3	−1	0	0
6	0	0	6	2	2	−2
5	−2	1	3	−1	0	0
5	−1	1	−2	4	−15	10
5	0	1	8	−6	5	5
5	1	1	−2	4	−15	10
5	2	1	3	−1	0	0
4	−2	2	6	−6	−4	−8
4	−1	2	−8	8	12	4
4	0	2	14	−14	−1	−7
4	1	2	−8	8	12	4
4	2	2	6	−6	−4	−8
3	−3	3	2	0	0	3
3	−2	3	6	0	−5	−11
3	−1	3	−6	0	40	1
3	0	3	16	0	−50	−6
3	1	3	−6	0	40	1
3	2	3	6	0	−5	−11
3	3	3	2	0	0	3
2	−2	4	6	6	−8	−4
2	−1	4	−8	−8	4	12
2	0	4	14	14	−7	−1
2	1	4	−8	−8	4	12
2	2	4	6	6	−8	−4
1	−2	5	3	1	1	−1
1	−1	5	−2	−4	−9	4
1	0	5	8	6	11	−1
1	1	5	−2	−4	−9	4
1	2	5	3	1	1	−1
0	0	6	6	−2	0	0
7	0	0	4	−2	6	−6
6	−2	1	6	0	−5	3
6	−1	1	0	0	10	−10
6	0	1	8	0	−20	24
6	1	1	0	0	10	−10
6	2	1	6	0	−5	3
5	−2	2	6	8	5	5
5	−1	2	−4	−12	15	−35
5	0	2	6	18	−15	35
5	1	2	−4	−12	15	−35
5	2	2	6	8	5	5
4	−3	3	−4	−4	−1	−3
4	−2	3	6	6	24	12
4	−1	3	−12	−12	−43	−49
4	0	3	10	10	70	50
4	1	3	−12	−12	−43	−49
4	2	3	6	6	24	12
4	3	3	−4	−4	−1	−3
3	−3	4	−4	4	−3	−1
3	−2	4	6	−6	12	24
3	−1	4	−12	12	−49	−43
3	0	4	10	−10	50	70
3	1	4	−12	12	−49	−43
3	2	4	6	−6	12	24
3	3	4	−4	4	−3	−1
2	−2	5	6	−8	−1	11
2	−1	5	−4	12	9	−29
2	0	5	6	−18	−21	41
2	1	5	−4	12	9	−29
2	2	5	6	−8	−1	11
1	−2	6	6	0	−5	3
1	−1	6	0	0	10	−10
1	0	6	8	0	−20	24
1	1	6	0	0	10	−10
1	2	6	6	0	−5	3
0	0	7	4	2	0	0

FIGURE 1. Fourier coefficients of $\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$ in the basic forms f_1, f_2, g_1, g_2, $a+c \le 7$

$$
\begin{aligned}
J^2 &= \\
&f_1^8 f_2^2 g_1^4 + 22 f_1^7 f_2^3 g_1^4 + 119 f_1^6 f_2^4 g_1^4 - 22 f_1^5 f_2^5 g_1^4 + f_1^4 f_2^6 g_1^4 + 2 f_1^8 f_2^2 g_1^3 g_2 + 46 f_1^7 f_2^3 g_1^3 g_2 + 282 f_1^6 f_2^4 g_1^3 g_2 \\
&+ 194 f_1^5 f_2^5 g_1^3 g_2 - 42 f_1^4 f_2^6 g_1^3 g_2 + 2 f_1^3 f_2^7 g_1^3 g_2 + f_1^8 f_2^2 g_1^2 g_2^2 + 22 f_1^7 f_2^3 g_1^2 g_2^2 + 120 f_1^6 f_2^4 g_1^2 g_2^2 + 120 f_1^4 f_2^6 g_1^2 g_2^2 \\
&- 22 f_1^3 f_2^7 g_1^2 g_2^2 + f_1^2 f_2^8 g_1^2 g_2^2 - 2 f_1^7 f_2^3 g_1 g_2^3 - 42 f_1^6 f_2^4 g_1 g_2^3 - 194 f_1^5 f_2^5 g_1 g_2^3 + 282 f_1^4 f_2^6 g_1 g_2^3 - 46 f_1^3 f_2^7 g_1 g_2^3 \\
&+ 2 f_1^2 f_2^8 g_1 g_2^3 + f_1^6 f_2^4 g_2^4 + 22 f_1^5 f_2^5 g_2^4 + 119 f_1^4 f_2^6 g_2^4 - 22 f_1^3 f_2^7 g_2^4 + f_1^2 f_2^8 g_2^4 - 2 f_1^7 f_2 g_1^5 - 110 f_1^6 f_2^2 g_1^5 \\
&- 166 f_1^5 f_2^3 g_1^5 - 12 f_1^4 f_2^4 g_1^5 - 4 f_1^7 f_2 g_1^4 g_2 - 242 f_1^6 f_2^2 g_1^4 g_2 - 450 f_1^5 f_2^3 g_1^4 g_2 - 438 f_1^4 f_2^4 g_1^4 g_2 - 24 f_1^3 f_2^5 g_1^4 g_2 \\
&- 2 f_1^7 f_2 g_1^3 g_2^2 - 142 f_1^6 f_2^2 g_1^3 g_2^2 + 62 f_1^5 f_2^3 g_1^3 g_2^2 + 110 f_1^4 f_2^4 g_1^3 g_2^2 - 370 f_1^3 f_2^5 g_1^3 g_2^2 - 10 f_1^2 f_2^6 g_1^3 g_2^2 - 10 f_1^6 f_2^2 g_1^2 g_2^3 \\
&+ 370 f_1^5 f_2^3 g_1^2 g_2^3 + 110 f_1^4 f_2^4 g_1^2 g_2^3 - 62 f_1^3 f_2^5 g_1^2 g_2^3 - 142 f_1^2 f_2^6 g_1^2 g_2^3 + 2 f_1 f_2^7 g_1^2 g_2^3 + 24 f_1^5 f_2^3 g_1 g_2^4 - 438 f_1^4 f_2^4 g_1 g_2^4 \\
&+ 450 f_1^3 f_2^5 g_1 g_2^4 - 242 f_1^2 f_2^6 g_1 g_2^4 + 4 f_1 f_2^7 g_1 g_2^4 - 12 f_1^4 f_2^4 g_2^5 + 166 f_1^3 f_2^5 g_2^5 - 110 f_1^2 f_2^6 g_2^5 + 2 f_1 f_2^7 g_2^5 + f_1^6 g_1^6 \\
&+ 152 f_1^5 f_2 g_1^6 + 48 f_1^4 f_2^2 g_1^6 + 2 f_1^6 g_1^5 g_2 + 340 f_1^5 f_2 g_1^5 g_2 + 300 f_1^4 f_2^2 g_1^5 g_2 + 96 f_1^3 f_2^3 g_1^5 g_2 + f_1^6 g_1^4 g_2^2 + 212 f_1^5 f_2 g_1^4 g_2^2 \\
&- 312 f_1^4 f_2^2 g_1^4 g_2^2 - 394 f_1^3 f_2^3 g_1^4 g_2^2 + 24 f_1^2 f_2^4 g_1^4 g_2^2 + 24 f_1^5 f_2 g_1^3 g_2^3 - 540 f_1^4 f_2^2 g_1^3 g_2^3 - 540 f_1^2 f_2^4 g_1^3 g_2^3 - 24 f_1 f_2^5 g_1^3 g_2^3 \\
&+ 24 f_1^4 f_2^2 g_1^2 g_2^4 + 394 f_1^3 f_2^3 g_1^2 g_2^4 - 312 f_1^2 f_2^4 g_1^2 g_2^4 - 212 f_1 f_2^5 g_1^2 g_2^4 + f_2^6 g_1^2 g_2^4 - 96 f_1^3 f_2^3 g_1 g_2^5 + 300 f_1^2 f_2^4 g_1 g_2^5 \\
&- 340 f_1 f_2^5 g_1 g_2^5 + 2 f_2^6 g_1 g_2^5 + 48 f_1^2 f_2^4 g_2^6 - 152 f_1 f_2^5 g_2^6 + f_2^6 g_2^6 - 64 f_1^4 g_1^7 - 144 f_1^4 g_1^6 g_2 - 128 f_1^3 f_2 g_1^6 g_2 \\
&- 92 f_1^4 g_1^5 g_2^2 + 264 f_1^3 f_2 g_1^5 g_2^2 + 32 f_1^2 f_2^2 g_1^5 g_2^2 - 12 f_1^4 g_1^4 g_2^3 + 296 f_1^3 f_2 g_1^4 g_2^3 + 4 f_1^2 f_2^2 g_1^4 g_2^3 + 96 f_1 f_2^3 g_1^4 g_2^3 \\
&- 96 f_1^3 f_2 g_1^3 g_2^4 + 4 f_1^2 f_2^2 g_1^3 g_2^4 - 296 f_1 f_2^3 g_1^3 g_2^4 - 12 f_2^4 g_1^3 g_2^4 + 32 f_1^2 f_2^2 g_1^2 g_2^5 - 264 f_1 f_2^3 g_1^2 g_2^5 - 92 f_2^4 g_1^2 g_2^5 \\
&+ 128 f_1 f_2^3 g_1 g_2^6 - 144 f_2^4 g_1 g_2^6 - 64 f_2^4 g_2^7 - 128 f_1^2 g_1^6 g_2^2 - 80 f_1^2 g_1^5 g_2^3 - 128 f_1 f_2 g_1^5 g_2^3 + 48 f_1^2 g_1^4 g_2^4 + 48 f_2^2 g_1^4 g_2^4 \\
&+ 128 f_1 f_2 g_1^3 g_2^5 - 80 f_2^2 g_1^3 g_2^5 - 128 f_2^2 g_1^2 g_2^6 - 64 g_1^5 g_2^4 - 64 g_1^4 g_2^5.
\end{aligned}
$$

FIGURE 2. Expression for J^2 in terms of the basic forms f_1, f_2, g_1, g_2

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