

THE RINGS OF HILBERT MODULAR FORMS FOR $\mathbb{Q}(\sqrt{29})$ AND $\mathbb{Q}(\sqrt{37})$

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ABSTRACT. We use Borcherds products and their restrictions to Hirzebruch-Zagier curves to determine generators and relations for the graded rings of Hilbert modular forms for the fields $\mathbb{Q}(\sqrt{29})$ and $\mathbb{Q}(\sqrt{37})$. These seem to be the first cases where the graded ring can be computed despite obstructions to the existence of Borcherds products with arbitrary divisors.

1. INTRODUCTION

The problem considered in this note is to determine generators and relations for the graded rings $M_*(\Gamma_K)$ of Hilbert modular forms over real quadratic fields K . There is some history to this. The structure has been worked out in the cases $\mathbb{Q}(\sqrt{5})$ (Gundlach, [12]), $\mathbb{Q}(\sqrt{2})$ (Hammond, [13]), $\mathbb{Q}(\sqrt{13})$ (van der Geer and Zagier, [10]), $\mathbb{Q}(\sqrt{17})$ (Hermann, [14] determined the structure of the subring of symmetric forms; the full ring appears in section 6.5 of [18]), and $\mathbb{Q}(\sqrt{6})$ (van der Geer, [8]). Section 6.2 of [18] gives a more detailed overview. Simpler derivations for the fields $\mathbb{Q}(\sqrt{5})$, $\mathbb{Q}(\sqrt{13})$, $\mathbb{Q}(\sqrt{17})$ were given by Mayer [18] using a reduction process against Borcherds products with simple zeros on the diagonal, and the work here extends this idea.

We only consider fields $K = \mathbb{Q}(\sqrt{p})$ of prime discriminant $p \equiv 1 \pmod{4}$ and class number one as this makes a number of aspects of the theory simpler (but there are ultimately workarounds for both of these assumptions). The first few such primes are $p = 5, 13, 17$, and the corresponding graded rings have the form

$$\begin{aligned} M_*(\Gamma_{\mathbb{Q}(\sqrt{5})}) &= \mathbb{C}[X_2, X_5, X_6, X_{15}]/(R_{30}), \\ M_*(\Gamma_{\mathbb{Q}(\sqrt{13})}) &= \mathbb{C}[X_2, X_3, X_4, X_5, X_6, X_8, X_9]/(R_9, R_{10}, R_{12}, R_{18}), \\ M_*(\Gamma_{\mathbb{Q}(\sqrt{17})}) &= \mathbb{C}[X_2, X_3, X_4^{(1)}, X_4^{(2)}, X_5^{(1)}, X_5^{(2)}, X_6^{(1)}, X_6^{(2)}, X_7, X_8, X_9]/\mathcal{I}, \\ \mathcal{I} &= (R_9, R_{10}, R_{11}, R_{12}^{(1)}, R_{12}^{(2)}, R_{13}, R_{14}, R_{18}), \end{aligned}$$

where $X_k, X_k^{(i)}$ denote modular forms of weight k and where $R_k, R_k^{(i)}$ denote relations that are homogeneous of weight k , see [18].

No results seem to have been written down for larger primes. We could expect this to be more difficult for primes $p > 17$, $p \equiv 1 \pmod{4}$ because the Weil representation associated to the finite quadratic module $(\mathcal{O}_K^\#/\mathcal{O}_K, N_{K/\mathbb{Q}})$ for the lattice of integers \mathcal{O}_K admits nonzero cusp forms of weight 2. More precisely, there are $\lceil \frac{p-5}{24} \rceil$ linearly independent cusp forms of weight 2 as is well-known (and originally due to Hecke). For one thing, these cusp forms lift injectively under the Doi-Naganuma map to holomorphic two-forms on the Hilbert modular surfaces. In particular those surfaces are never rational (see also [15], chapter 3 for the classification of Hilbert modular surfaces) so one roughly expects the rings in question to be more complicated. More immediately, these cusp forms act as obstructions to the possible divisors of Borcherds products in the sense of section 3.2 of [5] so it is generally not possible to find modular forms with simple divisors to reduce against.

Nevertheless, we are able to adapt the method of [18] to find complete results for the discriminants $p = 29, 37$:

Theorem 1. *The graded ring of Hilbert modular forms for the full ring of integers in $K = \mathbb{Q}(\sqrt{29})$ has the presentation*

$$M_*(\Gamma_K) = \mathbb{C}[X_2^{(1)}, X_2^{(2)}, X_3^{(1)}, X_3^{(2)}, X_4, X_5, X_6^{(1)}, X_6^{(2)}, X_7, X_8, X_9]/\mathcal{I}_{29},$$

where \mathcal{I}_{29} is a homogeneous ideal of relations of weight between 6 and 18.

Theorem 2. *The graded ring of Hilbert modular forms for the full ring of integers in $K = \mathbb{Q}(\sqrt{37})$ has the presentation*

$$M_*(\Gamma_K) = \mathbb{C}[X_2^{(1)}, X_2^{(2)}, X_3^{(1)}, X_3^{(2)}, X_3^{(3)}, X_4^{(1)}, X_4^{(2)}, X_5^{(1)}, X_5^{(2)}, X_6^{(1)}, X_6^{(2)}, X_6^{(3)}, X_7, X_8, X_9]/\mathcal{I}_{37},$$

where \mathcal{I}_{37} is also a homogeneous ideal of relations of weight between 6 and 18.

In both cases the generators are the weight two Eisenstein series and Borcherds lifts of modular functions. We found minimal systems of 35 generators of \mathcal{I}_{29} and 77 generators of \mathcal{I}_{37} . These are given explicitly in the ancillary material on arXiv and the author's website.

The rough idea of both proofs is to construct a Hilbert modular form (say, b) with a reasonably simple divisor. We then construct sufficiently many modular forms to be able to match any Hilbert modular form along Hirzebruch-Zagier curves to the orders specified by b . Any form which vanishes to those orders is divisible by b in $M_*(\Gamma_K)$ (by Koecher's principle) with quotient of strictly lesser weight, so we obtain an inductive argument which eventually determines the graded ring completely. The fact that the divisor of b is not usually irreducible makes the "matching" step somewhat delicate because the restrictions of a Hilbert modular form to Hirzebruch-Zagier curves interact at the points where those curves intersect. In the case $K = \mathbb{Q}(\sqrt{37})$ we also have to deal with a nontrivial character group.

The methods used here likely apply to some other real quadratic fields. Unfortunately the construction of generators is rather ad hoc, and we are unable to answer some natural questions. For example, are the graded rings always generated by theta lifts (Eisenstein series, Doi-Naganuma lifts and Borcherds products)? Also, if $K \neq \mathbb{Q}(\sqrt{5})$ then are the graded rings always presented by generators of weight at most 9 and by relations of weight at most 18? (The last question is similar to Main Theorem 1.4 of [20], which implies a statement of this type for all modular curves but which does not seem to apply to the Hilbert modular surfaces in an obvious way.)

There are computational difficulties involved in computing the Fourier expansion of a multivariate Borcherds product, even in the relatively simple case of Hilbert modular forms. (The paper [11] discusses this problem in general and offers some improvements over the naive method of computing.) The ideals of relations would be more difficult to work out directly. In this note we avoid this problem entirely by using Koecher's principle and the existence of certain Borcherds products with convenient divisors. In particular we never need to compute the Fourier expansion of any Hilbert modular form explicitly. We only use the interpretation of restrictions along various curves on the Hilbert modular surface as elliptic modular forms, which are easier to compute.

Acknowledgments. A large part of what is presented below is the result of computations in SAGE and Macaulay2. Some easier computations were carried out on a CoCalc server. The Magma implementation of Hilbert modular forms as described in [7] was useful for checking correctness. I thank Martin Raum for providing resources with which the more time-consuming computations were performed. I am also grateful to Jan Hendrik Bruinier and John Voight for helpful discussions. This work was supported by the LOEWE research unit Uniformized Structures in Arithmetic and Geometry.

2. HILBERT MODULAR FORMS

2.1. Hilbert modular forms. We quickly recall some of the basic theory of Hilbert modular forms attached to quadratic fields. The book [5] introduces Hilbert modular forms with a view toward Borcherds products and is therefore a useful reference for more details. Let \mathbb{H} be the usual upper half-plane and let K be a real quadratic field of discriminant d_K with ring of integers \mathcal{O}_K . For simplicity we assume that K has class number one. A **Hilbert modular form** of (parallel) weight $k \in \mathbb{N}_0$ is a holomorphic function of two variables $f : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{C}$ which satisfies

$$f\left(\frac{a\tau_1 + b}{c\tau_1 + d}, \frac{a'\tau_2 + b'}{c'\tau_2 + d'}\right) = (c\tau_1 + d)^k (c'\tau_2 + d')^k f(\tau_1, \tau_2)$$

for all $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_K = \mathrm{SL}_2(\mathcal{O}_K)$. Here a' denotes the conjugate of $a \in K$.

The invariance of f under all translations $T_b = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, $b \in \mathcal{O}_K$ implies that f has a Fourier series which we write in the form

$$f(\tau_1, \tau_2) = \sum_{\nu \in \mathcal{O}_K^\#} a(\nu) q_1^\nu q_2^{\nu'}, \quad q_i = e^{2\pi i \tau_i}, \quad a(\nu) \in \mathbb{C}.$$

Here $\mathcal{O}_K^\#$ is the codifferent

$$\mathcal{O}_K^\# = \{\nu \in K : \mathrm{Tr}_{K/\mathbb{Q}}(\nu x) \in \mathbb{Z} \text{ for all } x \in \mathcal{O}_K\} = \sqrt{1/d_K} \mathcal{O}_K.$$

The Götzky-Koecher principle ([5], theorem 1.20) states that $a(\nu) = 0$ whenever either $\nu < 0$ or $\nu' < 0$ so the usual boundedness condition at cusps for automorphic forms is automatically satisfied.

Some of the simplest Hilbert modular forms are the **Hecke Eisenstein series**

$$\mathbf{E}_k(\tau_1, \tau_2) = 1 + \frac{4}{\zeta_K(1-k)} \sum_{\substack{\nu \in \mathcal{O}_K^\# \\ \nu, \nu' > 0}} \sigma_{k-1}(\nu \sqrt{d_K} \mathcal{O}_K) q_1^\nu q_2^{\nu'}, \quad k \geq 2 \text{ even,}$$

where $\zeta_K(s)$ is the Dedekind zeta function of K and where $\sigma_{k-1}(\mathfrak{a}) = \sum_{\mathfrak{c}|\mathfrak{a}} N_{K/\mathbb{Q}}(\mathfrak{c})^{k-1}$ is the ideal divisor sum. When $k \geq 4$ these arise out of an averaging process similar to the Eisenstein series for $\mathrm{SL}_2(\mathbb{Z})$, and the case $k = 2$ can be recovered using Hecke's summation trick. See [5], section 1.5 for details.

Cusp forms are Hilbert modular forms that also vanish at the cusps: (∞, ∞) and (a, a') for all $a \in K$. Up to equivalence by Γ_K the number of cusps is exactly the (wide) class number of K . We have assumed that K has class number one and therefore a form $f(\tau_1, \tau_2) = \sum_{\nu} c(\nu) q_1^\nu q_2^{\nu'}$ is a cusp form if and only if $c(0) = 0$.

One can generalize Hilbert modular forms to include a character of $\mathrm{SL}_2(\mathcal{O}_K)$. The character group is always finite and depends up to isomorphism only on the ramification behavior of 2 and 3 in \mathcal{O}_K (see [3]). We will only consider **symmetric characters** χ : those for which $\chi(M) = \chi(M')$ for all $M \in \mathrm{SL}_2(\mathcal{O}_K)$. This is equivalent to $f(\tau_1, \tau_2)$ and $f^\sigma(\tau_1, \tau_2) = f(\tau_2, \tau_1)$ having the same character. Any Hilbert modular form f for a symmetric character can be decomposed into its symmetric and antisymmetric parts as

$$f = f^+ + f^-, \quad f^+ = \frac{f + f^\sigma}{2}, \quad f^- = \frac{f - f^\sigma}{2}.$$

(Here f is symmetric or antisymmetric if $f = f^\sigma$ or $f = -f^\sigma$, respectively.)

2.2. Graded rings of Hilbert modular forms. Let K be a real quadratic number field and set $\Gamma_K = \mathrm{SL}_2(\mathcal{O}_K)$. The set $M_{*,char}(\Gamma_K)$ of all Hilbert modular forms with all characters is naturally a graded ring where forms are graded by both their weights and characters. We denote by $M_{*,sym}(\Gamma_K)$ and $M_*(\Gamma_K)$ the subrings of Hilbert modular forms with symmetric and trivial characters, respectively.

The graded rings $M_*(\Gamma_K)$ are always finitely generated. In fact the Baily-Borel theorem (cf. [9], II.7) implies that their projective spectra $\mathrm{Proj} M_*(\Gamma_K)$ are complex surfaces. Together with the Noether normalization lemma this implies that we can always find a set of three algebraically independent forms f_1, f_2, f_3 such that $M_*(\Gamma_K)$ is finite over $\mathbb{C}[f_1, f_2, f_3]$ (i.e. finitely generated as a module). If f_1, f_2, f_3 have weights k_1, k_2, k_3 respectively then the theory of Hilbert series implies that there is a polynomial $p(t)$ for which

$$\sum_{k=0}^{\infty} \dim M_k(\Gamma_K) t^k = \frac{p(t)}{(1-t^{k_1})(1-t^{k_2})(1-t^{k_3})}.$$

Of course neither $\{f_1, f_2, f_3\}$ nor their weights $\{k_1, k_2, k_3\}$ are unique. Since $M_{*,char}(\Gamma_K)$ is finite over $M_*(\Gamma_K)$ (as some power of any Hilbert modular form will have trivial character), we get the same

result for the larger ring. In particular, for every character χ of Γ_K there is a polynomial $p_\chi(t)$ such that, with the same exponents k_1, k_2, k_3 ,

$$\sum_{k=0}^{\infty} \dim M_{k,\chi}(\Gamma_K) t^k = \frac{p_\chi(t)}{(1-t^{k_1})(1-t^{k_2})(1-t^{k_3})}.$$

Remark 1. It was proved in [19] (theorem 3.4) that the Hilbert series of even-weight modular forms for every real quadratic field except $\mathbb{Q}(\sqrt{5})$ has the form

$$\sum_{k=0}^{\infty} \dim M_{2k}(\Gamma_K) t^k = \frac{p(t)}{(1-t)^2(1-t^3)}$$

for an explicit polynomial $p(t)$ of degree exactly 6. This is not a special case of the paragraph above, and it does not hold for the full ring of modular forms.

2.3. Restrictions to Hirzebruch-Zagier curves. Hilbert modular forms can be restricted to the Hirzebruch-Zagier cycles of [16] to produce elliptic modular forms for $\Gamma_0(n)$. Recall that the Hirzebruch-Zagier cycle of discriminant n is the set $T_n \subseteq \mathbb{H} \times \mathbb{H}$ of all points (τ_1, τ_2) that satisfy an equation of the form

$$a\tau_1\tau_2 + \lambda\tau_1 + \lambda'\tau_2 + b = 0$$

for some $a, b \in \mathbb{Z}$ and $\lambda \in \mathcal{O}_K^\#$ with $ab - \lambda\lambda' = n/p$ (where $K = \mathbb{Q}(\sqrt{p})$). These cycles are Γ_K -invariant and there are inclusions $T_n \subseteq T_{nd^2}$ for all $n, d \in \mathbb{N}$ which are strict if $d > 1$.

Suppose $\lambda \in \mathcal{O}_K$ is a totally positive integer of norm $\ell = \lambda\lambda'$. Then the function

$$g(\tau) = \text{Res}_\lambda f(\tau) = f(\lambda\tau, \lambda'\tau)$$

satisfies

$$\begin{aligned} g(M \cdot \tau) &= f\left(\lambda \frac{a\tau + b}{\ell c\tau + d}, \lambda' \frac{a\tau + b}{\ell c\tau + d}\right) \\ &= f\left(\left(\begin{smallmatrix} a & \lambda b \\ \lambda' c & d \end{smallmatrix}\right) \cdot (\lambda\tau), \left(\begin{smallmatrix} a & \lambda' b \\ \lambda c & d \end{smallmatrix}\right) \cdot (\lambda'\tau)\right) \\ &= \chi\left(\left(\begin{smallmatrix} a & \lambda b \\ \lambda' c & d \end{smallmatrix}\right)\right) (\ell c\tau + d)^{2k} g(\tau) \end{aligned}$$

for all $M = \begin{pmatrix} a & b \\ \ell c & d \end{pmatrix} \in \Gamma_0(\ell)$. Moreover the growth condition at cusps follows from that of f and therefore $g(\tau)$ is a modular form of weight $2k$ and level $\Gamma_0(\ell)$ for the character

$$\chi_g\left(\left(\begin{smallmatrix} a & b \\ \ell c & d \end{smallmatrix}\right)\right) = \chi\left(\left(\begin{smallmatrix} a & \lambda b \\ \lambda' c & d \end{smallmatrix}\right)\right).$$

In addition, if ℓ is prime and f is symmetric, then we find

$$\begin{aligned} g(-1/\ell\tau) &= f\left(-\frac{\lambda}{\ell\tau}, -\frac{\lambda'}{\ell\tau}\right) \\ &= f\left(-\frac{1}{\lambda'\tau}, -\frac{1}{\lambda\tau}\right) \\ &= f\left(-\frac{1}{\lambda\tau}, -\frac{1}{\lambda'\tau}\right) \\ &= (\lambda\tau)^k (\lambda'\tau)^k f(\lambda\tau, \lambda'\tau) \\ &= \ell^k \tau^{2k} g(\tau), \end{aligned}$$

i.e. g is an eigenform of the Atkin-Lehner involution $W_\ell g(\tau) = \ell^{-k} \tau^{-2k} g(-1/\ell\tau)$ with eigenvalue $+1$. Similarly, if f is antisymmetric, then g will have eigenvalue -1 under W_ℓ .

Remark 2. If ℓ is a prime and λ is a fixed, totally positive integer of norm ℓ , then the Hirzebruch-Zagier cycle T_ℓ is exactly the orbit of

$$\{(\lambda\tau, \lambda'\tau) : \tau \in \mathbb{H}\} \cup \{(\lambda'\tau, \lambda\tau) : \tau \in \mathbb{H}\}$$

under $\text{SL}_2(\mathcal{O}_K)$. In that sense $f(\lambda\tau, \lambda'\tau)$ is a restriction of f to T_ℓ . In particular, if f is either symmetric or antisymmetric (which includes Borcherds products), then it vanishes along the entire

divisor T_ℓ whenever $f(\lambda\tau, \lambda'\tau) \equiv 0$ for any such element λ . Also note that T_1 is simply the orbit of the diagonal.

Remark 3. Even if the Hilbert modular form $f(\tau_1, \tau_2)$ vanishes along the diagonal, we can take its Taylor expansion in the variable τ_1 about the point τ_2 :

$$f(\tau_1, \tau_2) = g(\tau_2)(\tau_1 - \tau_2)^N + O((\tau_1 - \tau_2)^{N+1}), \quad |\tau_1 - \tau_2| < y_2,$$

for some function g . For any matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ compare

$$f(M \cdot \tau_1, M \cdot \tau_2) = (c\tau_1 + d)^{-N} (c\tau_2 + d)^{-N} g(M \cdot \tau_2)(\tau_1 - \tau_2)^N + O((\tau_1 - \tau_2)^{N+1})$$

with

$$(c\tau_1 + d)^k (c\tau_2 + d)^k f(M \cdot \tau_1, M \cdot \tau_2) = (c\tau_1 + d)^k (c\tau_2 + d)^k g(\tau_2)(\tau_1 - \tau_2)^N + O((\tau_1 - \tau_2)^{N+1})$$

to see that g is a modular form of weight $2k + 2N$. If $N \geq 1$, then $(\tau_1 - \tau_2)^N$ is not bounded, and therefore the growth condition on f forces g to be a cusp form. Similar expansions can be written out for Hilbert modular forms that vanish along the other Hirzebruch-Zagier curves, but we will not need any of these except for the diagonal.

2.4. Forced zeros. Modular forms for $\mathrm{SL}_2(\mathbb{Z})$ of certain weights have forced zeros at elliptic CM points. Namely, if the weight of a level one modular form f is not divisible by 4, then $f(i) = 0$; and if the weight is not divisible by 3, then $f(\rho) = 0$ where $\rho = e^{2\pi i/3}$. Some statements for Hilbert modular forms that are roughly in this spirit will be important later. We assume that $K = \mathbb{Q}(\sqrt{p})$ has prime discriminant p and fundamental unit ε .

Lemma 4. (i) *Every antisymmetric Hilbert modular form vanishes along the diagonal.*

(ii) *Every symmetric Hilbert modular form of weight k and character χ for which $\chi\left(\begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix}\right) \neq (-1)^k$ vanishes on the Hirzebruch-Zagier divisor T_p of discriminant p .*

(iii) *Every antisymmetric Hilbert modular form of weight k and character χ with $\chi\left(\begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix}\right) \neq (-1)^{k+1}$ vanishes on the Hirzebruch-Zagier divisor T_p of discriminant p .*

Proof. (i) $f(\tau_1, \tau_2) = -f(\tau_2, \tau_1)$ implies $f(\tau, \tau) = 0$ identically.

(ii) Since K has prime discriminant, the fundamental unit ε has negative norm. In particular $\lambda = \varepsilon\sqrt{p}$ is totally positive and T_p is the Γ_K -orbit of $\{(\lambda\tau, \lambda'\tau) : \tau \in \mathbb{H}\}$. We find

$$f(\lambda\tau, \lambda'\tau) = \chi\left(\begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix}\right) (-1)^k f(\lambda'\tau, \lambda\tau) = \chi\left(\begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix}\right) (-1)^k f(\lambda\tau, \lambda'\tau),$$

using the transformation under $\begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix}$ which sends $(\lambda'\tau, \lambda\tau)$ to $(\lambda\tau, \lambda'\tau)$ and using the symmetry of f . This implies the claim.

(iii) This uses nearly the same argument as (ii). □

3. TWO KINDS OF BORCHERDS PRODUCTS

3.1. Borchers products for $\Gamma_0(p)$. Let p be a prime. We recall a generalization of theorem 14.1 of [1] which produces modular products of level $\Gamma_0(p)$. (This is itself a special case of theorem 13.3 of [2].) Let A_p denote the Kohnen plus space of nearly-holomorphic modular forms of weight $1/2$ and level $(\Gamma_0(4p), \chi_\vartheta)$ with rational Fourier coefficients for the multiplier system χ_ϑ of the classical theta function $\vartheta(\tau) = 1 + 2q + 2q^4 + 2q^9 + \dots$. In other words A_p consists of Laurent series

$$f(q) = \sum_{n \gg -\infty} c(n)q^n, \quad q = e^{2\pi i\tau}, \quad c(n) \in \mathbb{Q}$$

for which $c(n) = 0$ if n is a quadratic nonresidue mod $4p$, and for which $f \cdot \vartheta$ is a modular form of weight 1 and level $\Gamma_0(4p)$ for the Dirichlet character

$$\chi(n) = \begin{cases} \left(\frac{-1}{n}\right) & : \quad n \text{ coprime to } 4p \\ 0 & : \quad \text{otherwise.} \end{cases}$$

Define

$$\delta_p(n) = \begin{cases} 1 & : \quad n \equiv 0 \pmod{p} \\ 1/2 & : \quad n \not\equiv 0 \pmod{p}, \end{cases}$$

and for $f(\tau) = \sum_n c(n)q^n$ let $\tilde{c}(n) = c(n)\delta_p(n)$. For any modular form $f \in A_p$ as above, Borcherds' theorem states that there is a rational number $h \in \mathbb{Q}$ (the "Weyl vector") such that

$$\Psi_f(\tau) = q^h \cdot \prod_{n=1}^{\infty} (1 - q^n)^{\tilde{c}(n^2)}$$

converges for large enough $\text{im}(\tau)$ and extends to a multivalued modular form of weight $c(0)$ and level $\Gamma_0(p)$ and some character. All zeros and singularities w of Ψ_f on \mathbb{H} satisfy a quadratic equation of the form $aw^2 + bw + c = 0$ with $a, b, c \in \mathbb{Z}$ and $a \equiv 0 \pmod{p}$, and the order of Ψ_f at such a point is a sum of coefficients for negative exponents in f :

$$\text{ord}(\Psi_f; w) = \sum_{m=1}^{\infty} \tilde{c}(m^2(b^2 - 4ac)).$$

In particular, Ψ_f has no branch cuts if these orders are all integral, and it is a holomorphic away from the cusps if these orders are also nonnegative. The Weyl vector h can be calculated as the constant term of the product of f with a mock Eisenstein series of weight $3/2$ and level $4p$ which is closely related to Zagier's Eisenstein series [22]. Of course one can also expand the product above and simply guess the (unique) value of h .

Example 5. Let $p = 3$; then A_3 contains the input function

$$f(\tau) = q^{-3} + 1 - 12q + 42q^4 - 76q^9 + 168q^{12} - 378q^{13} + 690q^{16} - 897q^{21} + 1456q^{24} - 3468q^{25} \pm \dots$$

which can be computed in SAGE to any desired precision by multiplying the weight four form

$$1 - 2q + q^3 - 12q^4 + 24q^5 + 44q^7 - 108q^8 - 2q^9 + 84q^{11} - 84q^{12} + O(q^{13}) \in M_4(\Gamma_0(12))$$

by the eta product $\frac{\eta(2\tau)}{\eta(\tau)^2\eta(12\tau)^6}$. Its lift Ψ_f is a modular form of weight 1 and level $(\Gamma_0(3), \chi)$ which has simple zeros exactly on the $\Gamma_0(3)$ -orbit of the CM point $w = \frac{1}{2} + \frac{\sqrt{3}}{6}i$ (i.e. the solution $w \in \mathbb{H}$ of $3w^2 - 3w + 1 = 0$). The Weyl vector is $h = 0$. From the Fourier expansion

$$\Psi_f(\tau) = (1 - q)^{-12/2}(1 - q^2)^{42/2}(1 - q^3)^{-76}(1 - q^4)^{690/2}(1 - q^5)^{-3468/2} \dots = 1 + 6q + 6q^3 + 6q^4 + O(q^6)$$

we see that Ψ_f is the theta function of $x^2 + xy + y^2$ and that χ is the nontrivial Nebentypus mod 3.

Remark 6. Borcherds products are more naturally orthogonal modular forms; in the special case above, one should really consider $\Gamma_0(p)$ acting by conjugation on a certain lattice of symmetric matrices and preserving the determinant (a quadratic form of signature $(1, 2)$). The Atkin-Lehner involution $W_p f(\tau) = p^{-k/2} \tau^{-k} f(-\frac{1}{p\tau})$ also preserves the determinant, and therefore all Borcherds products Ψ_f are eigenforms of W_p . We should also mention that passing to vector-valued input functions makes the condition that p is prime or even squarefree unnecessary.

3.2. Borcherds products for $\mathbb{Q}(\sqrt{p})$. We recall the construction of Hilbert modular forms by Borcherds products (using [2],[6]; see also section 3.2 of [5] for an introduction). Let $p \equiv 1 \pmod{4}$ be a prime discriminant and let $K = \mathbb{Q}(\sqrt{p})$ with ring of integers $\mathcal{O}_K = \mathbb{Z}[\frac{1+\sqrt{p}}{2}]$. One can essentially identify Hilbert modular forms of level \mathcal{O}_K with orthogonal modular forms for the signature $(2, 2)$ lattice $\mathcal{O}_K \oplus II_{1,1}$, where \mathcal{O}_K carries the norm-form $N_{K/\mathbb{Q}}$.

Let χ denote the quadratic Dirichlet character attached to K : $\chi(n) = \left(\frac{n}{p}\right)$, and let $M_0^!(\Gamma_0(p), \chi)$ denote the space of nearly-holomorphic modular forms of weight zero, level $\Gamma_0(p)$ and Nebentypus χ . Denote by B_p the plus-space

$$B_p = \left\{ F = \sum_{n \gg -\infty} c(n)q^n \in M_0^!(\Gamma_0(p), \chi) : c(n) \in \mathbb{Q}, c(n) = 0 \text{ if } \chi(n) = -1 \right\}.$$

As before we define $\delta_p(n) = 1$ if $n \equiv 0 \pmod{p}$ and $\delta_p(n) = 1/2$ otherwise, and set $\tilde{c}(n) = c(n)\delta_p(n)$ for any series $F = \sum_n c(n)q^n$.

Let $f \in B_p$. As in section 3.2 of [5], the principal part of F determines a splitting of $\mathbb{R}_{>0} \times \mathbb{R}_{>0}$ into **Weyl chambers** W . These are the connected components that are left after removing the curves

$\{(\lambda y, \lambda' y) : y \in \mathbb{R}_{>0}\}$ for all totally positive $\lambda \in \mathcal{O}_K$ with $c(-\lambda\lambda') \neq 0$. For any Weyl chamber W , Borcherds' theorem states that there is a number $h_W \in K$ (the ‘‘Weyl vector’’) such that

$$\Psi_F(\tau_1, \tau_2) = q_1^{h_W} q_2^{h'_W} \prod_{\substack{\nu \in \mathcal{O}_K^\# \\ \langle \nu, W \rangle > 0}} (1 - q_1^\nu q_2^{\nu'})^{\tilde{c}(p \cdot N_{K/\mathbb{Q}} \nu)}$$

converges for all $(\tau_1, \tau_2) \in \mathbb{H} \times \mathbb{H}$ with $\text{im}(\tau_1 \tau_2)$ sufficiently large and that Ψ_F extends to a multivalued Hilbert modular form of weight $c(0)/2$ for some character. Here $\langle \nu, W \rangle > 0$ means that $\text{Tr}_{K/\mathbb{Q}}(\nu w) > 0$ for all $w \in K$ with $(w, w') \in W$. The divisor of Ψ_F is a sum of Hirzebruch-Zagier cycles:

$$\text{div } \Psi_F = \sum_{n=1}^{\infty} \tilde{c}(-n) T_n.$$

In particular, Ψ_F is holomorphic if and only if all orders $\sum_{m=1}^{\infty} \tilde{c}(-nm^2)$ are nonnegative integral. (Note that this does not imply that all $c(-n)$ are nonnegative.)

Remark 7. The Borcherds products attached to the various Weyl chambers and a single input function all have the same weight and divisor so the quotient between any two is a nonzero constant (by the Götzkky-Koecher principle). That constant is not generally 1. From this it follows that the Borcherds products attached to the different Weyl chambers all have the same character as well.

Remark 8. A holomorphic Borcherds product Ψ_F is a cusp form if and only if the compactification of its divisor intersects all cusps. This too can be read off the input function. When K has class number one, it is sufficient for the input function f to have a nonzero coefficient in any exponent q^{-n} with $n \in N_{K/\mathbb{Q}} \mathcal{O}_K$. Therefore Hilbert modular form Borcherds products of small weight tend to be cusp forms. Indeed all Borcherds products we construct in this note for $\mathbb{Q}(\sqrt{29})$ and $\mathbb{Q}(\sqrt{37})$ are cusp forms.

3.3. Restricting Borcherds products. Let $K = \mathbb{Q}(\sqrt{p})$ for a prime $p \equiv 1 \pmod{4}$. Suppose $\Psi_F(\tau_1, \tau_2) = q_1^h q_2^{h'} \prod_{\langle \nu, W \rangle > 0} (1 - q_1^\nu q_2^{\nu'})^{\tilde{c}(p \cdot N_{K/\mathbb{Q}} \nu)}$ is the expansion of the Borcherds lift of

$$F(\tau) = \sum_{n \in \mathbb{Z}} c(n) q^n$$

within some Weyl chamber W , and set $\tilde{c}(n) = c(n)$ or $c(n)/2$ as in section 3.2. Also suppose that $\lambda \in \mathcal{O}_K$ satisfies $(\lambda, \lambda') \in W$ (in particular, λ is totally positive) and that $N_{K/\mathbb{Q}}(\lambda) = \ell$ is 1 or a prime. Restricting gives

$$\Psi_F(\lambda\tau, \lambda'\tau) = q^{\text{Tr}(\lambda h)} \prod_{n=1}^{\infty} (1 - q^n)^{\tilde{b}(n^2)}$$

for the coefficients

$$\tilde{b}(n^2) = \sum_{\substack{\langle \nu, W \rangle > 0 \\ \text{Tr}(\lambda\nu) = n}} \tilde{c}(p \cdot N_{K/\mathbb{Q}} \nu) = \sum_{\text{Tr}(\lambda\nu) = n} \tilde{c}(p \cdot N_{K/\mathbb{Q}} \nu),$$

where we use the fact that $\text{Tr}(\lambda\nu) > 0$ implies $\langle \nu, W \rangle > 0$, by lemma 3.2 of [4]. This suggests that $\text{Res}_\lambda \Psi_F(\tau) = \Psi_F(\lambda\tau, \lambda'\tau)$ itself is likely a Borcherds product as in section 3.1. This is true and for $\ell \neq p$ the input function can be produced from F as follows. Divide the coefficients of all q^n , $n \not\equiv 0 \pmod{p}$ in F by two (to get the modified coefficients $\tilde{c}(n)$), change variables $q \mapsto q^{4\ell/p}$, multiply the result by $\vartheta(\tau/p)$ where $\vartheta(\tau) = 1 + 2q + 2q^4 + 2q^9 + \dots$ is the theta function, and restrict to integer exponents. In other words we take the input function

$$f(\tau) = \sum_{n=0}^{\infty} \sum_{\substack{r \in \mathbb{Z} \\ 4\ell n + r^2 \equiv 0 \pmod{p}}} \tilde{c}(n) q^{(4\ell n + r^2)/p} = \sum_{m=0}^{\infty} b(m) q^m, \quad b(m) = \sum_{\substack{r \in \mathbb{Z} \\ mp - r^2 \equiv 0 \pmod{4\ell}}} \tilde{c}\left(\frac{mp - r^2}{4\ell}\right).$$

The procedure for constructing f (and in particular the proof that $f \in A_\ell$ is a valid input function in the sense of section 3.1) is the same as remark 10 and example 11 of [21] with minor changes. That is, the vector-valued modular form corresponding to F is transformed into a weak Jacobi form $\varphi(\tau, z)$ of fractional index $\frac{p}{4\ell}$ through the theta decomposition and then $f(\tau)$ is the Kohnen plus form

corresponding to the vector-valued modular form $\varphi(\tau, 0)$. This is ultimately the same procedure as the theta-contraction of [17].

The square coefficients are

$$b(n^2) = \sum_{\substack{r \in \mathbb{Z} \\ pn^2 - r^2 \equiv 0 \pmod{4\ell}}} \tilde{c}(p \cdot N_{K/\mathbb{Q}}\nu), \quad \nu = \frac{n - r/\sqrt{p}}{2\lambda}.$$

Note that if $n \not\equiv 0 \pmod{\ell}$ and $pn^2 - r^2 \equiv 0 \pmod{4\ell}$, then exactly one of $\sqrt{p}n \pm r$ is divisible by 2λ in \mathcal{O}_K (and the other is divisible by $2\lambda'$) because λ is prime and $\lambda \nmid \gcd(\sqrt{p}n + r, \sqrt{p}n - r)$. (Remember that we assume \mathcal{O}_K has class number one.) If $n \equiv 0 \pmod{\ell}$ and therefore also $r \equiv 0 \pmod{\ell}$, then both $\sqrt{p}n \pm r$ are divisible by 2ℓ and therefore by 2λ . Altogether it follows that the $\tilde{b}(n^2) = b(n^2)\delta_p(n)$ satisfy

$$\tilde{b}(n^2) = \sum_{\substack{\nu \in \mathcal{O}_K^\# \\ \text{Tr}_{K/\mathbb{Q}}(\lambda\nu) = n}} \tilde{c}(p \cdot N_{K/\mathbb{Q}}\nu) = \begin{cases} b(n^2)/2 : n \not\equiv 0 \pmod{p} \\ b(n^2) : n \equiv 0 \pmod{p}, \end{cases}$$

and therefore that $\text{Res}_\lambda \Psi_F(\tau) = \Psi_f(\tau)$ holds. (That the Weyl vectors match up follows from the fact that $\text{Res}_\lambda \Psi_F(\tau)$ and $\Psi_f(\tau)$ are meromorphic modular forms of the same weight and level whose Fourier expansions are equal up to a shift; so their quotient is a power of $q = e^{2\pi i\tau}$ which is invariant under $\text{SL}_2(\mathbb{Z})$ and therefore equals 1. It does not seem to be as easy to show that the Weyl vectors match directly from their definitions.)

When $\ell = p$ the procedure of theta contraction is similar. The only difference in the result is that we do not divide any of the coefficients of F by 2; instead we multiply $F(4\tau)\vartheta(\tau/p)$ directly and restrict to integer exponents.

Example 9. Let $p = 5$. Then B_5 contains the input function

$$F(\tau) = 2q^{-1} + 10 + 22q - 108q^4 + 110q^5 + 88q^6 - 790q^9 + 680q^{10} \pm \dots$$

which lifts to Gundlach's weight 5 cusp form $s_5(\tau_1, \tau_2)$ for $\mathbb{Q}(\sqrt{5})$ that vanishes exactly on the Γ_K -orbit of the diagonal [12]. There is a Weyl chamber of F with Weyl vector $h = \frac{1}{2} - \frac{\sqrt{5}}{10}$. We compute the restriction to the Hirzebruch-Zagier curve T_5 as follows. Take the totally positive element $\lambda_5 = \frac{5+\sqrt{5}}{2}$ of norm 5, which satisfies $\text{Tr}_{K/\mathbb{Q}}(\lambda_5 h) > 0$. We multiply

$$\begin{aligned} & F(4\tau)\vartheta(\tau/5) \\ &= \left(2q^{-4} + 10 + 22q^4 - 108q^{16} + 110q^{20} + 88q^{24} \pm \dots\right) \left(1 + 2q^{1/5} + 2q^{4/5} + 2q^{9/5} + \dots\right) \\ &= 2q^{-4} + 4q^{-19/5} + 4q^{-16/5} + 4q^{-11/5} + 4q^{-4/5} + 10 + 20q^{1/5} + 20q^{4/5} + 4q + \dots \end{aligned}$$

and restrict to integer exponents to find the input function

$$f(\tau) = 2q^{-4} + 10 + 4q + 22q^4 + 20q^5 + 44q^9 - 104q^{16} + 130q^{20} \pm \dots \in A_5.$$

Its lift is

$$\begin{aligned} s_5(\lambda_5\tau, \lambda_5'\tau) &= q^{\text{Tr}_{K/\mathbb{Q}}(\lambda_5 h)} (1-q)^2 (1-q^2)^{11} (1-q^3)^{22} (1-q^4)^{-52} \dots \\ &= q^2 (1-2q-10q^2+140q^4+\dots) \\ &= \frac{1}{4} \eta(\tau)^8 \eta(5\tau)^8 (5E_2(5\tau) - E_2(\tau)), \end{aligned}$$

where $E_2(\tau) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n)q^n$ with $\sigma_1(n) = \sum_{d|n} d$, and $\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1-q^n)$. In particular, the case of Borcherds' theorem in section 3.1 shows that this has simple zeros exactly on the $\Gamma_0(5)$ -orbit of the discriminant 4 CM point $\tau = \frac{2+i}{5}$, so $(\lambda_5\tau, \lambda_5'\tau)$ represents the intersection $T_1 \cap T_5$.

To compute the restriction onto the curve T_{11} with $\lambda_{11} = \frac{7+\sqrt{5}}{2}$, we have to pass from the coefficients $c(n)$ to $\tilde{c}(n)$. Therefore we multiply

$$\begin{aligned} & \left(q^{-44/5} + 10 + 11q^{44/5} - 54q^{176/5} + 110q^{44} + 44q^{264/5} \pm \dots \right) \left(1 + 2q^{1/5} + 2q^{4/5} + 2q^{9/5} + \dots \right) \\ & = q^{-44/5} + 2q^{-43/5} + 2q^{-8} + 2q^{-7} + 2q^{-28/5} + 2q^{-19/5} + 2q^{-8/5} + 10 + 20q^{1/5} + 20q^{4/5} + 2q + \dots \end{aligned}$$

and restrict to integer exponents to find the input form

$$f(\tau) = 2q^{-8} + 2q^{-7} + 10 + 2q + 2q^4 + 20q^5 + 22q^9 + 22q^{12} + \dots \in A_{11}$$

whose Borcherds lift is

$$\begin{aligned} s_5(\lambda_{11}\tau, \lambda'_{11}\tau) & = q^{\text{Tr}_{K/\mathbb{Q}}(\lambda_{11}h)}(1-q)(1-q^2)(1-q^3)^{11} \dots \\ & = q^3(1-q-q^2-10q^3+10q^5+121q^6 \pm \dots) \in S_{10}(\Gamma_0(11)). \end{aligned}$$

Note that λ_5 and λ_{11} lie in the same Weyl chamber of F so there really is a single Hilbert modular form with these restrictions. In general this would only hold up to a constant multiple as in remark 7.

4. THE GRADED RING OF HILBERT MODULAR FORMS FOR $\mathbb{Q}(\sqrt{29})$

Let $K = \mathbb{Q}(\sqrt{29})$ with ring of integers $\mathcal{O}_K = \mathbb{Z}[\frac{1+\sqrt{29}}{2}]$. Since 2 and 3 both remain inert in \mathcal{O}_K , there are no nontrivial characters of $\text{SL}_2(\mathcal{O}_K)$ (by [3]).

The principal parts that extend to nearly-holomorphic forms in the input space B_{29} can be determined by theorem 3.49 of [5]; essentially the only obstruction is that the product with the unique cusp form in the plus space

$$q - 3q^4 - 3q^5 + 5q^6 + 2q^7 - 2q^9 - q^{13} - q^{16} \pm \dots \in S_2(\Gamma_0(29), \chi)$$

has constant term zero. In this way one can prove that there exist nearly-holomorphic modular forms $F_2, F_3, G_3, F_4, F_6, G_6$ of weight 0 and level $\Gamma_0(29)$ and Nebentypus $\chi(d) = \left(\frac{29}{d}\right)$ whose Fourier expansions begin as follows:

$$\begin{aligned} F_2(\tau) & = 2q^{-4} + 6q^{-1} + 4 + 2q - 2q^4 + 8q^5 - 2q^6 + 14q^7 + 4q^9 - 10q^{13} \pm \dots \\ F_3(\tau) & = 2q^{-6} + 4q^{-4} + 2q^{-1} + 6 - 6q - 2q^4 + 12q^5 + 16q^6 + 18q^7 + 28q^9 - 10q^{13} \pm \dots \\ G_3(\tau) & = 2q^{-5} + 6q^{-1} + 6 + 10q + 4q^4 + 12q^5 + 0q^6 + 0q^7 - 6q^9 + 20q^{13} \pm \dots \\ F_4(\tau) & = 2q^{-6} + 2q^{-5} + 2q^{-4} + 2q^{-1} + 8 + 2q + 4q^4 + 16q^5 + 18q^6 + 4q^7 + 18q^9 + 20q^{13} \pm \dots \\ F_6(\tau) & = q^{-29} + 6q^{-1} + 12 + 30q + 124q^4 + 162q^5 + 252q^6 + 336q^7 + 648q^9 + 2050q^{13} + \dots \\ G_6(\tau) & = 2q^{-16} + 2q^{-6} + 2q^{-5} - 2q^{-1} + 12 - 10q - 34q^4 - 48q^5 - 70q^6 + 80q^7 + 186q^9 + 336q^{13} \pm \dots \end{aligned}$$

The Borcherds lifts $\phi_i = \Psi_{F_i}$, $\psi_i = \Psi_{G_i}$ have weight i , and their divisors can be read off the principal parts of the input functions:

$$\text{div } \phi_2 = 3T_1 + T_4, \quad \text{div } \phi_3 = T_1 + 2T_4 + T_6, \quad \text{div } \psi_3 = 3T_1 + T_5,$$

$$\text{div } \phi_4 = T_1 + T_4 + T_5 + T_6, \quad \text{div } \phi_6 = 3T_1 + T_{29}, \quad \text{div } \psi_6 = -T_1 + T_5 + T_6 + T_{16}.$$

We will also need the following (holomorphic) quotients of the above forms as generators:

$$\begin{aligned} \phi_5 & = \frac{\psi_3\phi_4}{\phi_2}, \quad \text{div } \phi_5 = T_1 + 2T_5 + T_6; \\ \phi_7 & = \frac{\phi_3\phi_6}{\phi_2}, \quad \text{div } \phi_7 = T_1 + T_4 + T_6 + T_{29}; \\ \phi_8 & = \frac{\phi_3\psi_3\phi_6}{\phi_2^2}, \quad \text{div } \phi_8 = T_1 + T_5 + T_6 + T_{29}; \\ \phi_9 & = \frac{\phi_3^2\psi_3\phi_6}{\phi_2^3}, \quad \text{div } \phi_9 = -T_1 + T_4 + T_5 + 2T_6 + T_{29}. \end{aligned}$$

All are cusp forms. We fix $\lambda_1 = 1$ and $\lambda_5 = \frac{7+\sqrt{29}}{2}$ and compute restrictions up to a constant multiple (which does not matter here and is omitted by abuse of notation) using the procedure in section 3.3:

$$\begin{aligned}\text{Res}_{\lambda_1} \psi_6(\tau) &= \Delta(\tau) = q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 \pm \dots \\ \text{Res}_{\lambda_1} \phi_9(\tau) &= \Delta(\tau)E_6(\tau) = q - 528q^2 - 4284q^3 + 147712q^4 \pm \dots \\ \text{Res}_{\lambda_1} \mathbf{E}_2(\tau) &= E_4(\tau) = 1 + 240q + 2160q^2 + 6720q^3 + 17520q^4 + \dots\end{aligned}$$

(Note that these require no computation because the respective spaces of modular forms or cusp forms are one-dimensional. One can read off the divisors that the restrictions of the Borcherds products are nonzero, and the Eisenstein series has a nonzero constant term.) Also,

$$\begin{aligned}\text{Res}_{\lambda_5} \phi_2(\tau) &= s_4(\tau) = q - 4q^2 + 2q^3 + 8q^4 - 5q^5 \pm \dots \\ \text{Res}_{\lambda_5} \phi_3(\tau) &= s_4(\tau)e_2(\tau) = q + 2q^2 - 4q^3 - 28q^4 + 25q^5 \pm \dots \\ \text{Res}_{\lambda_5} \phi_4(\tau) &= s_4(\tau)^2 = q^2 - 8q^3 + 20q^4 - 70q^6 \pm \dots \\ \text{Res}_{\lambda_5} \phi_6(\tau) &= s_4(\tau)^2 e_4(\tau) = q^2 - 18q^3 + 10q^4 + 240q^5 \pm \dots \\ \text{Res}_{\lambda_5} \phi_7(\tau) &= e_2(\tau)e_4(\tau)s_4(\tau)^2 = q^2 - 12q^3 - 80q^4 + 870q^6 \pm \dots \\ \text{Res}_{\lambda_5} \mathbf{E}_2(\tau) &= e_2(\tau)^2 - 4s_4(\tau) = 1 + 8q + 88q^2 + 256q^3 + 664q^4 + 1400q^5 + \dots\end{aligned}$$

with the omitted restrictions above being zero, where we have fixed the following generators of the ring of modular forms for $\Gamma_0(5)$:

$$\begin{aligned}e_2(\tau) &= \frac{5E_2(5\tau) - E_2(\tau)}{4} = 1 + 6q + 18q^2 + 24q^3 + 42q^4 + 6q^5 + \dots \\ e_4(\tau) &= \frac{25E_4(5\tau) - E_5(\tau)}{24} = 1 - 10q - 90q^2 - 280q^3 - 730q^4 - 1010q^5 - \dots \\ s_4(\tau) &= \eta(\tau)^4 \eta(5\tau)^4 = q - 4q^2 + 2q^3 + 8q^4 - 5q^5 \pm \dots\end{aligned}$$

which satisfy a single relation

$$e_4^2 = e_2^4 - 44s_4e_2^2 - 16s_4^2$$

in weight 8. Note that e_2, e_4 are eigenforms of the Atkin-Lehner involution W_5 with eigenvalue -1 while s_4 has eigenvalue $+1$ under W_5 .

Lemma 10. *Let f be a Hilbert modular form of parallel weight for $\mathbb{Q}(\sqrt{29})$ and let $\lambda_5 = \frac{7+\sqrt{29}}{2}$. Then the restriction $f(\lambda_5\tau, \lambda_5'\tau)$ of f to T_5 coincides with the restriction of some polynomial in $\mathbf{E}_2, \phi_2, \phi_3, \phi_6, \phi_7$.*

Proof. Any form f can be split into its symmetric and antisymmetric parts, and it is enough to argue for each of those parts separately. We also argue differently depending on whether the weight is even or odd.

Case 1: suppose $f \in M_k^{sym}(\Gamma_K)$ is symmetric and has even weight. Then $\text{Res}_{\lambda_5} f$ is an eigenform of W_5 with eigenvalue $+1$ and has weight divisible by 4 and is therefore a polynomial expression P in e_2^2, s_4 . Since

$$e_2^2 = \text{Res}_{\lambda_5}(\mathbf{E}_2 + 4\phi_2), \quad s_4 = \text{Res}_{\lambda_5} \phi_2$$

we find

$$\text{Res}_{\lambda_5} f = \text{Res}_{\lambda_5} P(\mathbf{E}_2 + 4\phi_2, \phi_2) = \text{Res}_{\lambda_5} \tilde{P}(\mathbf{E}_2, \phi_2)$$

for some polynomial \tilde{P} .

Case 2: suppose $f \in M_k^{anti}(\Gamma_K)$ is antisymmetric and has odd weight. In particular f vanishes on the diagonal and is therefore a cusp form; so $\text{Res}_{\lambda_5} f$ is also a cusp form (and is therefore divisible by s_4) and has weight $2 \pmod{4}$ (and is therefore divisible by e_2). The remainder $\frac{\text{Res}_{\lambda_5} f}{e_2 s_4}$ has eigenvalue $+1$ under the involution W_5 and is therefore a polynomial in e_2^2, s_4 . Since

$$e_2 s_4 = \text{Res}_{\lambda_5} \phi_3$$

it follows from case 1 that $\text{Res}_{\lambda_5} f$ coincides with the restriction of some polynomial in $\mathbf{E}_2, \phi_2, \phi_3$.

Case 3: suppose $f \in M_k^{\text{sym}}(\Gamma_K)$ is symmetric and has odd weight. In particular it has a forced zero on T_{29} . After multiplying by ϕ_2 , we obtain a form $\phi_2 f$ which vanishes on T_{29} and vanishes to order at least three along the diagonal and is therefore divisible by ϕ_6 . The remainder $\frac{f\phi_2}{\phi_6}$ is antisymmetric of odd weight so by the argument in case 2 we find

$$\text{Res}_{\lambda_5}(f\phi_2\phi_6^{-1}) = e_2 s_4 P(e_2^2, s_4)$$

for some polynomial P , and therefore

$$\text{Res}_{\lambda_5} f = \frac{\text{Res}_{\lambda_5} \phi_6}{\text{Res}_{\lambda_5} \phi_2} \cdot e_2 s_4 P(e_2^2, s_4) = e_2 e_4 s_4^2 P(e_2^2, s_4).$$

Since $e_2 e_4 s_4^2 = \text{Res}_{\lambda_5} \phi_7$ and e_2^2, s_4 were accounted for in case 1, we see that $\text{Res}_{\lambda_5} f$ is the restriction of some polynomial in $\mathbf{E}_2, \phi_2, \phi_7$.

Case 4: suppose $f \in M_k^{\text{anti}}(\Gamma_K)$ is antisymmetric and has even weight; in this case there are forced zeros on both T_1 and T_{29} . After multiplying by ϕ_3 we see that $\phi_3 f$ is divisible by ϕ_6 , and the remainder is antisymmetric of odd weight. By the argument in case 2 again we find

$$\text{Res}_{\lambda_5}(f\phi_3\phi_6^{-1}) = e_2 s_4 P(e_2^2, s_4)$$

for some polynomial P , and therefore

$$\text{Res}_{\lambda_5} f = \frac{\text{Res}_{\lambda_5} \phi_6}{\text{Res}_{\lambda_5} \phi_3} \cdot e_2 s_4 P(e_2^2, s_4) = e_4 s_4^2 P(e_2^2, s_4).$$

Since $e_4 s_4^2 = \text{Res}_{\lambda_5} \phi_6$ we see that $\text{Res}_{\lambda_5} f$ is the restriction of some polynomial in $\mathbf{E}_2, \phi_2, \phi_6$. \square

Proposition 11. *Every Hilbert modular form of parallel weight for $\mathbb{Q}(\sqrt{29})$ is an isobaric polynomial in the generators*

$$\mathbf{E}_2, \phi_2, \phi_3, \psi_3, \phi_4, \phi_5, \phi_6, \psi_6, \phi_7, \phi_8, \phi_9.$$

Proof. We use induction on the weight k of f . If $k \leq 0$, then f is constant.

In general, we can assume by the previous lemma that f vanishes on T_5 . In particular, f is a cusp form. As before, we split f into its symmetric and antisymmetric parts and obtain four cases to distinguish.

Case 1: k is odd and f is symmetric. In particular f has a forced zero on T_{29} . Then the restriction of f to the diagonal is a cusp form of weight $2k \equiv 2(4)$ and therefore has the form

$$f(\tau, \tau) = \Delta(\tau) E_6(\tau) P(E_4, \Delta)$$

for some polynomial P ; so $g := f - \phi_9 P(\mathbf{E}_2, \psi_6)$ vanishes on both T_1 and T_{29} . Actually g must have at least a double zero along T_1 because $g \cdot \phi_4$ is divisible by ϕ_6 and the quotient $\frac{g\phi_4}{\phi_6}$ is antisymmetric and therefore vanishes on the diagonal. Therefore, g expands about the diagonal in the form

$$g(\tau_1, \tau_2) = h(\tau_2)(\tau_1 - \tau_2)^2 + O((\tau_1 - \tau_2)^3),$$

where h is a cusp form of weight $2k + 4 \equiv 2(4)$ and therefore has the form

$$h(\tau) = \Delta(\tau) E_6(\tau) P_2(E_4, \Delta)$$

for some polynomial P_2 . On the other hand, the form ϕ_7 has a double zero on T_1 and the expansion

$$\phi_7(\tau_1, \tau_2) = \Delta(\tau_2) E_6(\tau_2)(\tau_1 - \tau_2)^2 + O((\tau_1 - \tau_2)^3)$$

up to a constant multiple (since ΔE_6 is the unique normalized cusp form of weight 18); for the argument here there is no loss of generality in assuming that constant is 1. Therefore

$$g - \phi_7 P_2(\mathbf{E}_2, \psi_6) = f - \phi_9 P(\mathbf{E}_2, \psi_6) - \phi_7 P_2(\mathbf{E}_2, \psi_6)$$

has a zero on T_{29} and at least a triple zero on T_1 and is therefore divisible by ϕ_6 . The quotient

$$\frac{f - \phi_9 P(\mathbf{E}_2, \psi_6) - \phi_7 P_2(\mathbf{E}_2, \psi_6)}{\phi_6}$$

has smaller weight than f , and the claim follows by induction.

Case 2: k is odd and f is antisymmetric. Expand f about the diagonal in the form

$$f(\tau_1, \tau_2) = h(\tau_2)(\tau_1 - \tau_2) + O((\tau_1 - \tau_2)^2);$$

then $h(\tau_2)$ is a cusp form of weight $2k + 2 \equiv 0 \pmod{4}$ and therefore of the form $\Delta(\tau)P(E_4, \Delta)$ for some polynomial P . Since ϕ_5 has the expansion

$$\phi_5(\tau_1, \tau_2) = \Delta(\tau_2)(\tau_1 - \tau_2) + O((\tau_1 - \tau_2)^2)$$

(up to a possible constant multiple, $\Delta(\tau_2)$ is the unique cusp form of weight 12), it follows that

$$g = f - \phi_5 P(\mathbf{E}_2, \psi_6)$$

has at least a double zero along T_1 and still vanishes along T_5 (because ϕ_5 does). Actually g has at least a triple zero along T_1 because $g\phi_5$ is divisible by ψ_3 with the result being antisymmetric and therefore still vanishing along T_1 . It follows that g itself is divisible by ψ_3 . The quotient $\frac{g}{\psi_3}$ has smaller weight than f , and the claim follows by induction.

Case 3: k is even and f is symmetric. The restriction $\text{Res}_{\lambda_1} f$ has the form $\Delta(\tau)P(E_4, \Delta)$ for some polynomial P and therefore $g = f - \psi_6 P(\mathbf{E}_2, \psi_6)$ vanishes along T_1 (and it still vanishes along T_5 , because both f and ψ_6 do). Actually it must have at least a double zero along T_1 , because $g\phi_4$ is divisible by ψ_3 with the result being antisymmetric and therefore still vanishing along T_1 . We expand g along the diagonal as

$$g(\tau_1, \tau_2) = h(\tau_2)(\tau_1 - \tau_2)^2 + O((\tau_1 - \tau_2)^3);$$

then h is a cusp form of weight divisible by 4 so it is $h = \Delta \cdot P_2(E_4, \Delta)$ for another polynomial P_2 . With the expansion

$$\phi_4(\tau_1, \tau_2) = \Delta(\tau_2)(\tau_1 - \tau_2)^2 + O((\tau_1 - \tau_2)^3)$$

(up to a constant multiple), it follows that $g - \phi_4 P_2(\mathbf{E}_2, \psi_6)$ has at least a triple zero along T_1 and continues to vanish along T_5 (since both g and ϕ_4 do). Therefore $g - \phi_4 P_2(\mathbf{E}_2, \psi_6)$ is divisible by ψ_3 with the quotient having smaller weight, and the claim follows by induction.

Case 4: k is even and f is antisymmetric (and therefore f vanishes on both T_1 and T_{29}). Expand f about the diagonal:

$$f(\tau_1, \tau_2) = h(\tau_2)(\tau_1 - \tau_2) + O((\tau_1 - \tau_2)^2).$$

Then h is a cusp form of weight $2 \pmod{4}$ and therefore $h = \Delta E_6 \cdot P(E_4, \Delta)$ for some polynomial P . Since

$$\phi_8(\tau_1, \tau_2) = \Delta(\tau_2)E_6(\tau_2)(\tau_1 - \tau_2) + O((\tau_1 - \tau_2)^2)$$

(up to a constant multiple), it follows that $f - \phi_8 P(\mathbf{E}_2, \psi_6)$ has at least a double zero along T_1 (and continues to vanish on T_5 , because ϕ_8 does). As in the previous cases it follows that $f - \phi_8 P(\mathbf{E}_2, \psi_6)$ actually has at least a triple zero along T_1 . It is therefore divisible by ψ_3 with the result having smaller weight, and the claim follows by induction. □

Theorem 1. *The graded ring $M_*(\Gamma_{\mathbb{Q}(\sqrt{29})})$ is defined by the 11 generators*

$$\mathbf{E}_2, \phi_2, \phi_3, \psi_3, \phi_4, \phi_5, \phi_6, \psi_6, \phi_7, \phi_8, \phi_9$$

in weights 2, 2, 3, 3, 4, 5, 6, 6, 7, 8, 9 and by 35 relations in weights 6 through 18.

Proof. The strategy we used to compute relations is as follows. There are totally positive elements

$$\lambda_{23} = \frac{11 + \sqrt{29}}{2}, \quad \lambda_{35} = \frac{13 + \sqrt{29}}{2}$$

of norms 23 and 35, respectively, which lie in the same Weyl chamber with respect to all 10 of the Borcherds products considered above. Moreover the obstruction principle shows that there is a nearly-holomorphic modular form whose Fourier expansion begins $2q^{-35} + 2q^{-23} + 48 + \dots$ and which lifts to a Borcherds product ϕ_{24} of weight 24 with simple zeros along T_{23} and T_{35} . Any nonzero Hilbert modular form f whose restrictions to T_{23} and T_{35} both vanish is divisible by ϕ_{24} ; the quotient $\frac{f}{\phi_{24}}$ is

also holomorphic in the cusps by the Götzky-Koecher principle and therefore has nonnegative weight so f must have had weight at least 24.

In this way we are able to compute all relations up to weight 24 by computing relations only among the restrictions to T_{23} and T_{35} which is somewhat easier. The exact relations we found are tedious to write out and are left to the ancillary material. We mention here that $\{\mathbf{E}_2, \phi_3, \psi_3\}$ is algebraically independent (because the restrictions of \mathbf{E}_2 and ϕ_3 to T_5 are already algebraically independent, and ψ_3 vanishes there) and that any relation can be reduced against the relations we found up to weight 18 to eliminate the variables other than $\mathbf{E}_2, \phi_3, \psi_3$. It follows that the 35 relations we found up to weight 18 generate the entire ideal. \square

Corollary 12. *The Hilbert series is*

$$\sum_{k=0}^{\infty} \dim M_k(\Gamma_{\mathbb{Q}(\sqrt{29})}) t^k = \frac{1-t+t^2+t^4+t^6-t^7+t^8}{(1-t)(1-t^2)(1-t^3)}.$$

5. THE GRADED RING OF HILBERT MODULAR FORMS FOR $\mathbb{Q}(\sqrt{37})$

Let $K = \mathbb{Q}(\sqrt{37})$ with ring of integers $\mathcal{O}_K = \mathbb{Z}[\omega]$, $\omega = \frac{1+\sqrt{37}}{2}$. Since 2 is inert and 3 splits in \mathcal{O}_K , theorem 1 of [3] implies that the full character group of $\mathrm{SL}_2(\mathcal{O}_K)$ is generated by two characters χ_1, χ_ω which are characterized by their effect on the translations T_1, T_ω by 1 and ω :

$$\chi_1(T_1) = e^{2\pi i/3}, \quad \chi_1(T_\omega) = 1$$

and

$$\chi_\omega(T_1) = 1, \quad \chi_\omega(T_\omega) = e^{2\pi i/3}.$$

The condition for a character χ being symmetric is $\chi(T_\omega) = \chi(T_{\omega'}) = \chi(T_1)\chi(T_\omega^{-1})$, i.e. $\chi(T_1) = \chi(T_\omega)^2$. In particular the group of symmetric characters of $\mathrm{SL}_2(\mathcal{O}_K)$ is cyclic of order three and generated by $\chi = \chi_1\chi_\omega^{-1}$.

Our procedure is very similar to the previous section (although the role of the Hirzebruch-Zagier curve T_5 above is played instead by T_3). The only additional difficulty comes from the fact that Borcherds products here may have a nontrivial character.

We compute input functions using the obstruction principle as in the last section. There is again a single cusp form that acts as the obstruction to extending arbitrary principal parts to nearly-holomorphic modular forms and it is

$$q - q^3 - 2q^4 + 3q^7 - 2q^9 + 4q^{10} - 3q^{11} + 2q^{12} \pm \dots \in S_2(\Gamma_0(37), \chi).$$

Therefore there exist nearly-holomorphic modular forms $F_1, G_1, F_2, G_2, F_4, G_7$ of weight 0 and level $\Gamma_0(37)$ and Nebentypus $\chi(d) = \left(\frac{37}{d}\right)$ whose Fourier expansions begin as follows:

$$F_1(\tau) = 2q^{-3} + 2q^{-1} + 2 + 2q + 4q^3 + 2q^7 + 0q^9 - 4q^{10} + 2q^{11} + 4q^{12} \pm \dots$$

$$G_1(\tau) = 2q^{-4} + 4q^{-1} + 2 - 2q + 2q^3 - 2q^4 + 6q^7 + 8q^9 + 2q^{10} + 2q^{11} - 4q^{12} \pm \dots$$

$$F_2(\tau) = 2q^{-10} + 2q^{-4} + 2q^{-3} - 2q^{-1} + 4 - 6q - 8q^3 + 8q^4 + 18q^7 - 4q^9 + 6q^{10} + 34q^{11} + 30q^{12} \pm \dots$$

$$G_2(\tau) = 2q^{-10} + 4q^{-4} + 4 - 10q - 10q^3 + 6q^4 + 22q^7 + 4q^9 + 12q^{10} + 34q^{11} + 22q^{12} \pm \dots$$

$$F_4(\tau) = q^{-37} + 2q^{-1} + 8 + 24q + 42q^3 + 80q^4 + 170q^7 + 300q^9 + 416q^{10} + 504q^{11} + 664q^{12} + \dots$$

$$G_7(\tau) = q^{-37} + 2q^{-12} + 2q^{-3} + 14 + 28q + 60q^3 + 68q^4 + 150q^7 + 308q^9 + 450q^{10} + 520q^{11} + 700q^{12} + \dots$$

The Borcherds lifts $\phi_i = \Psi_{F_i}$, $\psi_i = \Psi_{G_i}$ have weight i and their divisors can be read off the principal parts of the input functions:

$$\mathrm{div} \phi_1 = T_1 + T_3, \quad \mathrm{div} \psi_1 = 2T_1 + T_4, \quad \mathrm{div} \phi_2 = -T_1 + T_3 + T_4 + T_{10},$$

$$\mathrm{div} \psi_2 = 2T_4 + T_{10}, \quad \mathrm{div} \phi_4 = T_1 + T_{37}, \quad \mathrm{div} \psi_7 = T_3 + T_{12} + T_{37}.$$

In particular $\phi_1, \psi_1, \psi_2, \phi_4$ vanish on the diagonal and ϕ_2, ψ_7 do not. Moreover ϕ_1, ψ_1, ϕ_4 are anti-symmetric and ϕ_2, ψ_2, ψ_7 are symmetric, and all of the products above have a nontrivial character of order 3; the notation is chosen such that ϕ_1, ϕ_2, ϕ_4 have the same character χ and that ψ_1, ψ_2, ψ_7 have the same character χ^{-1} . All of this can be proved directly, but it is easier to read it off of the

restrictions to the diagonal and to the curve T_3 which are worked out in the next paragraph. Note that by construction $G_2 = F_2 + G_1 - F_1$ and therefore $\phi_1\psi_2 = \psi_1\phi_2$.

We fix the totally positive integers $\lambda_1 = 1$, $\lambda_3 = \frac{7+\sqrt{37}}{2}$. Then one can compute the restrictions

$$\text{Res}_{\lambda_1}\phi_2(\tau) = \eta(\tau)^8, \quad \text{Res}_{\lambda_1}\psi_7(\tau) = E_6(\tau)\eta(\tau)^{16}, \quad \text{Res}_{\lambda_1}\mathbf{E}_2(\tau) = E_4(\tau)$$

and

$$\text{Res}_{\lambda_3}\psi_1(\tau) = \eta(\tau)^2\eta(3\tau)^2 = q^{1/3}(1 - 2q - q^2 + 5q^4 + 4q^5 - 7q^6 \pm \dots)$$

$$\text{Res}_{\lambda_3}\psi_2(\tau) = \eta(\tau)^2\eta(3\tau)^2e_2(\tau) = q^{1/3}(1 + 10q + 11q^2 - 72q^3 + 29q^4 - 44q^5 + 29q^6 \pm \dots)$$

$$\text{Res}_{\lambda_3}\phi_4(\tau) = \eta(\tau)^4\eta(3\tau)^4e_4(\tau) = q^{2/3}(1 - 34q - 148q^2 + 454q^3 - 559q^4 + 2418q^5 + 680q^6 \pm \dots)$$

$$\text{Res}_{\lambda_3}\mathbf{E}_2(\tau) = e_2(\tau)^2 = 1 + 24q + 216q^2 + 888q^3 + 1752q^4 + 3024q^5 + 7992q^6 \pm \dots$$

with the omitted restrictions above being zero. Here we use the notation

$$e_2(\tau) = \frac{3E_2(3\tau) - E_2(\tau)}{2} = 1 + 12q + 36q^2 + \dots, \quad e_4(\tau) = \frac{9E_4(3\tau) - E_4(\tau)}{8} = 1 - 30q - 270q^2 - \dots$$

Using these computations we can prove:

Lemma 13. *Let $f \in M_*(\Gamma_K)$ be a Hilbert modular form of parallel weight and trivial character. Then there is a polynomial P such that $f - P(\psi_1, \psi_2, \phi_4, \mathbf{E}_2)$ vanishes on the Hirzebruch-Zagier divisor T_3 . If f is symmetric or antisymmetric, then one can also choose $P(\psi_1, \psi_2, \phi_4, \mathbf{E}_2)$ to be symmetric or antisymmetric, respectively.*

Proof. The ring of modular forms of level $\Gamma_0(3)$ and trivial character is generated by the forms

$$e_2(\tau) = \frac{3E_2(3\tau) - E_2(\tau)}{2} = 1 + 12q + 36q^2 + 12q^3 + \dots$$

$$e_4(\tau) = \frac{9E_4(3\tau) - E_4(\tau)}{8} = 1 - 30q - 270q^2 - 570q^3 - \dots$$

$$s_6(\tau) = \eta(\tau)^6\eta(3\tau)^6 = q - 6q^2 + 9q^3 + 4q^4 \pm \dots$$

where e_2 has weight 2, e_4 has weight 4, s_6 is a cusp form of weight 6, together with a single relation

$$e_4^2 = e_2^4 - 108e_2s_6$$

in weight 8. All three generators are eigenforms of the Atkin-Lehner operator W_3 with eigenvalue -1 . We split f into its symmetric and antisymmetric parts and argue for each part separately.

Case 1: suppose $f \in M_k^{sym}(\Gamma_K)$ is symmetric and has even weight. Then $\text{Res}_{\lambda_3}f$ is an eigenform of W_3 with eigenvalue $+1$ and has weight divisible by 4; so it is a polynomial expression in e_2^2, e_2s_6, s_6^2 . Since

$$e_2^2 = \text{Res}_{\lambda_3}\mathbf{E}_2, \quad e_2s_6 = \text{Res}_{\lambda_3}(\psi_1^2\psi_2), \quad s_6^2 = \text{Res}_{\lambda_3}\psi_1^6,$$

we can find a polynomial in $\psi_1^6, \psi_1^2\psi_2, \mathbf{E}_2$ whose restriction to T_3 equals $\text{Res}_{\lambda_3}f$. This will also be symmetric because $\psi_1^2, \psi_2, \mathbf{E}_2$ are symmetric.

Case 2: suppose $f \in M_k^{sym}(\Gamma_K)$ is symmetric and has odd weight. Then f is a cusp form and therefore $\text{Res}_{\lambda_3}f$ is also a cusp form; in particular it is divisible by s_6 , and $s_6^{-1}\text{Res}_{\lambda_3}f$ has eigenvalue -1 under W_3 . Therefore all monomials in e_2, e_4, s_6 that occur in $s_6^{-1}\text{Res}_{\lambda_3}f$ must contain e_4 (as all expressions involving only e_2, s_6 in weights $0 \pmod{4}$ will have eigenvalue $+1$) so $\text{Res}_{\lambda_3}f$ is e_4s_6 multiplied by some polynomial in e_2^2, e_2s_6, s_6^2 . The claim follows from the previous case together with

$$e_4s_6 = \text{Res}_{\lambda_3}(\phi_4\psi_1),$$

and the resulting polynomial expression will again be symmetric because $\phi_4\psi_1$ is symmetric.

Case 3: suppose $f \in M_k^{anti}(\Gamma_K)$ is antisymmetric. In particular f vanishes along the diagonal and therefore also at the cusps; so $\text{Res}_{\lambda_3}f$ is a cusp form that is an eigenform of W_3 with eigenvalue -1 . After dividing by s_6 , we reduce to case 1; since $s_6 = \text{Res}_{\lambda_3}\psi_1^3$ and ψ_1^3 is antisymmetric, the resulting polynomial expression is antisymmetric. \square

This quickly implies the same result for all of the symmetric characters:

Lemma 14. *Let $f \in M_*(\Gamma_K, \chi)$ be a Hilbert modular form of any weight with symmetric character χ . Then there is a polynomial P as above for which $f - P(\psi_1, \psi_2, \phi_4, \mathbf{E}_2)$ vanishes on the divisor T_3 .*

Proof. The nontrivial symmetric characters restrict on T_3 to the characters of $s_2(\tau) = \eta(\tau)^2\eta(3\tau)^2$ and its square. Note that any modular form of level $\Gamma_0(3)$ that transforms under one of those characters is a cusp form (because the order at all cusps is nonintegral and in particular nonzero) and is therefore divisible by s_2 or s_2^2 , respectively. Since $\text{Res}_{\lambda_3}\psi_1 = s_2$, we can use the previous argument. \square

By further restricting to the diagonal we can compute generators:

Proposition 15. *Every Hilbert modular form $f \in M_*(\Gamma_K, \chi)$ for a symmetric character χ is an isobaric polynomial in $\phi_1, \psi_1, \phi_2, \psi_2, \phi_4, \psi_7, \mathbf{E}_2$.*

Proof. We induct on the weight of f . If f has weight zero, then it is constant. Otherwise, we again split f into its symmetric and antisymmetric parts and argue for each part separately.

Case 1: suppose f is symmetric and has odd weight. Then f has a forced zero on T_{37} so its restriction $g = \text{Res}_{\lambda_1}f$ to the diagonal is a cusp form for $\text{SL}_2(\mathbb{Z})$ of weight $2 \pmod 4$ for some cubic character, and is therefore some polynomial expression P in $E_4, \eta^8, E_6\eta^{16}$. More precisely, if g has the character of η^8 then $\eta^{-8}g$ has weight $0 \pmod 4$ and no character, and is therefore a polynomial in E_4 and $\Delta = (\eta^8)^3$. If g has the character of η^{16} then $\eta^{-16}g$ has weight $2 \pmod 4$ and no character, and therefore is also divisible by E_6 , and $E_6^{-1}\eta^{-16}g$ is holomorphic of weight $0 \pmod 4$ so the previous sentence applies. Finally, if g has the trivial character then it vanishes at i and is therefore divisible by E_6 . As g is a cusp form, we can divide by $\Delta E_6 = \eta^8 \cdot E_6\eta^{16}$ with the remainder having weight $0 \pmod 4$ again. Altogether, we can write

$$\text{Res}_{\lambda_1}f = P(E_4, \eta^8, E_6\eta^{16}) = \text{Res}_{\lambda_1}P(\mathbf{E}_2, \phi_2, \psi_7).$$

In particular, $f - P(\mathbf{E}_2, \phi_2, \psi_7)$ vanishes on T_1 and is still symmetric and of odd weight so it continues to vanish on T_{37} . Therefore we can divide by ϕ_4 (which has simple zeros only on T_1 and T_{37}) to obtain a holomorphic form $\frac{f - P(\mathbf{E}_2, \phi_2, \psi_7)}{\phi_4}$ of lower weight which is a polynomial as in the claim by induction. This implies the claim for f itself.

Case 2: suppose f is antisymmetric and has odd weight. We use the previous lemmas and assume without loss of generality that f already vanishes on T_3 . As an antisymmetric form, f has a forced zero on T_1 . In particular it is divisible by ϕ_1 and $\frac{f}{\phi_1}$ has smaller weight than f so the claim follows by induction.

Case 3: Now suppose that f has even weight; and again by the previous lemmas, assume without loss of generality that f already vanishes on T_3 . Then f is a cusp form, and its restriction to the diagonal is a cusp form of level 1 whose weight is divisible by 4 and which is therefore some polynomial in E_4, η^8 in which all monomials contain η^8 . In particular, $\text{Res}_{\lambda_1}f$ is the restriction of some polynomial $Q(\mathbf{E}_2, \phi_2)$ in which all monomials contain ϕ_2 . Since ϕ_2 vanishes on T_3 , we find that $f - Q(\mathbf{E}_2, \phi_2)$ vanishes on both T_1 and T_3 , so it is divisible by ϕ_1 ; as $\frac{f - Q(\mathbf{E}_2, \phi_2)}{\phi_1}$ has weight strictly less than that of f , the claim follows by induction. \square

Proposition 16. *The graded ring $M_{*,\text{sym}}(\Gamma_K)$ of modular forms with symmetric characters is presented by seven generators*

$$\phi_1, \psi_1, \mathbf{E}_2, \phi_2, \psi_2, \phi_4, \psi_7$$

of weights 1, 1, 2, 2, 2, 4, 7 and by 9 relations $R_{3,1}, R_{4,1}, R_{4,\chi}, R_{8,1}, R_{8,\chi}, R_{8,\chi^2}, R_{9,\chi}, R_{11,1}, R_{14,\chi}$ in weights 3 through 14.

Proof. We take the positive-definite integers

$$\lambda_{21} = \frac{11 + \sqrt{37}}{2}, \quad \lambda_{33} = \frac{13 + \sqrt{37}}{2}$$

of norm 21 and 33 which lie in a common Weyl chamber for each of the Borcherds products above. The obstruction principle shows that there is a nearly-holomorphic modular form in B_{37} whose Fourier expansion begins $2q^{-33} + 2q^{-21} + 32 + \dots$ and which therefore lifts to a Hilbert modular form f_{16} of weight 16 with simple zeros exactly along T_{21} and T_{33} . By the Götzkky-Koecher principle (arguing as in the previous section) every nonzero modular form which vanishes along both T_{21} and T_{33} must have weight at least 16 so we are able to find all relations up to weight 16 by only computing relations among the restrictions to T_{21}, T_{33} .

In this way we compute the relations

$$\begin{aligned}
R_{3,1} &= \psi_1\phi_2 - \phi_1\psi_2; \\
R_{4,1} &= 5\phi_2\psi_2 - 5\phi_1\psi_1\mathbf{E}_2 - 4\phi_1^2\psi_1^2 - 20\phi_1^2\phi_2; \\
R_{4,\chi} &= 5\psi_2^2 - 5\psi_1^2\mathbf{E}_2 - 4\phi_1\psi_1^3 - 20\phi_1^2\psi_2; \\
R_{8,1} &= 5\phi_2^2\phi_4 - 5\phi_1^2\mathbf{E}_2\phi_4 - 20\phi_1\psi_7 + 36\phi_1^3\psi_1\phi_4; \\
R_{8,\chi} &= \psi_1\psi_7 - \phi_1^2\phi_2\phi_4 - 2\phi_1^2\psi_1^2\phi_4; \\
R_{8,\chi^2} &= 125\phi_4^2 - 125\psi_1^4\mathbf{E}_2^2 + 13500\psi_1^6\psi_2 - 250\phi_1\psi_1\psi_2\mathbf{E}_2^2 + 38050\phi_1\psi_1^5\mathbf{E}_2 - 125\phi_1^2\mathbf{E}_2^3 + 27000\phi_1^2\phi_2^3 \\
&\quad + 35850\phi_1^2\psi_1^2\psi_2\mathbf{E}_2 + 33895\phi_1^2\psi_1^6 + 11700\phi_1^3\psi_1\mathbf{E}_2^2 + 163340\phi_1^3\psi_1^3\psi_2 + 42750\phi_1^4\phi_2\mathbf{E}_2 \\
&\quad + 119710\phi_1^4\psi_1^2\mathbf{E}_2 + 82466\phi_1^5\psi_1^3 + 447700\phi_1^6\psi_2 + 3375\phi_1^8; \\
R_{9,\chi} &= 5\psi_2\psi_7 - 5\psi_1^3\mathbf{E}_2\phi_4 - 20\phi_1^2\psi_7 - 10\phi_1^2\psi_1\psi_2\phi_4 + 36\phi_1^4\psi_1\phi_4; \\
R_{11,1} &= 500\phi_4\psi_7 - 125\phi_1\phi_2^2\mathbf{E}_2^3 + 27000\phi_1\phi_2^5 - 1000\phi_1^2\psi_1^5\mathbf{E}_2^2 + 108000\phi_1^2\psi_1^7\psi_2 + 125\phi_1^3\mathbf{E}_2^4 \\
&\quad + 15750\phi_1^3\phi_2^3\mathbf{E}_2 - 2500\phi_1^3\psi_1^2\psi_2\mathbf{E}_2^2 + 358400\phi_1^3\psi_1^6\mathbf{E}_2 - 1900\phi_1^4\psi_1\mathbf{E}_2^3 + 439000\phi_1^4\psi_1^3\psi_2\mathbf{E}_2 \\
&\quad + 314360\phi_1^4\psi_1^7 + 50\phi_1^5\phi_2\mathbf{E}_2^2 + 236200\phi_1^5\psi_1^2\mathbf{E}_2^2 + 1658300\phi_1^5\psi_1^4\psi_2 + 1725760\phi_1^6\psi_1^3\mathbf{E}_2 \\
&\quad + 1554640\phi_1^7\psi_2\mathbf{E}_2 + 2571775\phi_1^7\phi_2^2 + 1182416\phi_1^7\psi_1^4 + 6680424\phi_1^8\psi_1\psi_2 - 3375\phi_1^9\mathbf{E}_2 + 24300\phi_1^{10}\psi_1; \\
R_{14,\chi} &= 10000\psi_7^2 - 625\phi_2^4\mathbf{E}_2^3 + 135000\phi_2^7 + 1250\phi_1^2\phi_2^2\mathbf{E}_2^4 - 56250\phi_1^2\phi_2^5\mathbf{E}_2 - 625\phi_1^4\mathbf{E}_2^5 - 78500\phi_1^4\phi_2^3\mathbf{E}_2^2 \\
&\quad - 40000\phi_1^4\psi_1^6\mathbf{E}_2^2 + 4320000\phi_1^4\psi_1^8\psi_2 - 120000\phi_1^5\psi_1^3\psi_2\mathbf{E}_2^2 + 16496000\phi_1^5\psi_1^7\mathbf{E}_2 - 56250\phi_1^6\phi_2\mathbf{E}_2^3 \\
&\quad + 16746875\phi_1^6\phi_2^4 - 129600\phi_1^6\psi_1^2\mathbf{E}_2^3 + 24728000\phi_1^6\psi_1^4\psi_2\mathbf{E}_2 + 14302400\phi_1^6\psi_1^8 + 18188000\phi_1^7\psi_1^3\mathbf{E}_2^2 \\
&\quad + 81259200\phi_1^7\psi_1^5\psi_2 + 6064800\phi_1^8\psi_2\mathbf{E}_2^2 + 24373050\phi_1^8\phi_2^2\mathbf{E}_2 + 109220400\phi_1^8\psi_1^4\mathbf{E}_2 \\
&\quad + 133829600\phi_1^9\psi_1\psi_2\mathbf{E}_2 + 73829440\phi_1^9\psi_1^5 + 16875\phi_1^{10}\mathbf{E}_2^2 + 424151360\phi_1^{10}\psi_1^2\psi_2 \\
&\quad + 229181280\phi_1^{11}\psi_1\mathbf{E}_2 + 917697120\phi_1^{12}\phi_2 + 184414224\phi_1^{12}\psi_1^2.
\end{aligned}$$

Any relation of weight greater than 16 can be reduced against the relations above to eliminate the variables $\phi_2, \psi_2, \phi_4, \psi_7$, and the remaining generators $\phi_1, \psi_1, \mathbf{E}_2$ are algebraically independent (because the restrictions of ψ_1, \mathbf{E}_2 to T_3 are already algebraically independent, and ϕ_1 vanishes along T_3). Therefore the relations above are enough. \square

Corollary 17. *Let $\mathbf{e}_1, \mathbf{e}_\chi, \mathbf{e}_{\chi^2}$ be multiplicative symbols which represent the symmetric characters of $\mathrm{SL}_2(\mathcal{O}_K)$. Then we have the Hilbert series*

$$\sum_{k=0}^{\infty} \sum_{i \in \mathbb{Z}/3\mathbb{Z}} \dim M_{k,\chi^i}(\Gamma_K) t^k \mathbf{e}_{\chi^i} = \frac{1 + t^2(\mathbf{e}_\chi + \mathbf{e}_{\chi^2}) - t^3 + t^4(\mathbf{e}_\chi + \mathbf{e}_{\chi^2}) - t^5\mathbf{e}_\chi + 2t^6 + t^6\mathbf{e}_{\chi^2} - t^7\mathbf{e}_\chi}{(1 - t\mathbf{e}_\chi)(1 - t\mathbf{e}_{\chi^2})(1 - t^2)}.$$

With a bit of algebra (e.g. replacing $\frac{1}{1-t\mathbf{e}_\chi}$ by $\frac{1+t\mathbf{e}_\chi+t^2\mathbf{e}_{\chi^2}}{1-t^3}$) it follows that the dimensions of Hilbert modular forms with trivial character have the generating series

$$\sum_{k=0}^{\infty} \dim M_k(\Gamma_K) t^k = \frac{1 - t + t^2 + t^3 + t^4 + t^5 + t^6 - t^7 + t^8}{(1-t)(1-t^2)(1-t^3)}.$$

Proof. This can be computed quickly in Macaulay2, for example. The exponents and characters in the denominator come from $\phi_1, \psi_1, \mathbf{E}_2$. \square

Theorem 2. *The graded ring of modular forms $M_*(\Gamma_K)$ is presented by the 15 generators*

$$\mathbf{E}_2, \phi_1\psi_1, \phi_1^3, \phi_1\psi_2, \psi_1^3, \phi_2\psi_2, \psi_1^2\psi_2, \psi_1\phi_4, \phi_1\phi_2^2, \phi_2^3, \psi_2\phi_4, \phi_1^2\phi_4, \phi_1\phi_2\phi_4, \phi_2^2\phi_4, \phi_2\psi_7$$

of weights 2, 2, 3, 3, 3, 4, 4, 5, 5, 6, 6, 6, 7, 8, 9 and by 77 relations in weights 6 through 18.

A number of the 77 defining relations are those that are implied by the notation, e.g. $(\phi_1\psi_1)^3 = \phi_1^3 \cdot \psi_1^3$.

Proof. As generators we can take the monomials in $\mathbf{E}_2, \phi_1, \psi_1, \phi_2, \psi_2, \phi_4, \psi_7$ that have trivial character (and that cannot be further split into monomials with trivial character). Many of these monomials turn out not to be necessary due to relations such as $\psi_1\phi_2 = \phi_1\psi_2$. The 15 given above are minimal in that sense.

The ideal of relations is given by intersecting the ideal of relations from the previous lemma with the subring of Hilbert modular forms with trivial character. Finding this intersection and minimal generators for it is a straightforward Gröbner basis computation which was done in Macaulay2. We leave the explicit relations to the supplementary material. \square

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