On Hermitian Eisenstein Series of Degree 2

by

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Abstract. We consider the Hermitian Eisenstein series $E_k^{(\mathbb{K})}$ of degree 2 and weight k associated with an imaginary-quadratic number field $\mathbb{K}$ and determine the influence of $\mathbb{K}$ on the arithmetic and the growth of its Fourier coefficients. We find that they satisfy the identity ${E_4^{(\mathbb{K})}}^2 = E_8^{(\mathbb{K})}$, which is well-known for Siegel modular forms of degree 2, if and only if $\mathbb{K} = \mathbb{Q}(\sqrt{-3})$. As an application, we show that the Eisenstein series $E_k^{(\mathbb{K})}$, $k = 4, 6, 8, 10, 12$ are algebraically independent whenever $\mathbb{K} \neq \mathbb{Q}(\sqrt{-3})$. The difference between the Siegel and the restriction of the Hermitian to the Siegel half-space is a cusp form in the Maaß space that does not vanish identically for sufficiently large weight; however, when the weight is fixed, we will see that it tends to 0 as the discriminant tends to $-\infty$. Finally, we show that these forms generate the space of cusp forms in the Maaß Spezialschar as a module over the Hecke algebra as $\mathbb{K}$ varies over imaginary-quadratic number fields.

Keywords: Hermitian Eisenstein series, Siegel Eisenstein series, Maaß space

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1 Introduction

Eisenstein series are the most common examples of modular forms in several variables. In the case of Hermitian modular forms associated with an imaginary-quadratic number field $\mathbb{K}$, they were introduced by H. Braun [2]. In this paper we consider Hermitian Eisenstein series of degree 2. Its Fourier expansion is determined by the Maaß condition and has been worked out explicitly (cf. [16], [10]).

This knowledge leads to new insights on the influence of $\mathbb{K}$ on the arithmetic and the growth of the Fourier coefficients. We will demonstrate that the Eisenstein series $E_k^{(\mathbb{K})}$ satisfy the identity ${E_4^{(\mathbb{K})}}^2 = E_8^{(\mathbb{K})}$, whose analogue for Siegel modular forms of degree 2 is well known, if and only if $\mathbb{K} = \mathbb{Q}(\sqrt{-3})$. This allows us to show that the Eisenstein series $E_k^{(\mathbb{K})}$, $k = 4, 6, 8, 10, 12$, are algebraically independent whenever $\mathbb{K} \neq \mathbb{Q}(\sqrt{-3})$.

Finally we consider the difference between $E_k^{(\mathbb{K})}$ restricted to the Siegel half-space and the Siegel Eisenstein series of weight k. This is a Siegel cusp form in the Maaß space. When the weight k is fixed, its limit is 0 as the discriminant of $\mathbb{K}$ tends to $-\infty$. On the other hand, it does not vanish identically whenever the weight is sufficiently large. Moreover the vanishing of the above difference can be characterized by the vanishing of a Shimura lift. This allows us to show that the subspace of cusp forms in the Maaß space is generated by these restrictions as a module over the Hecke algebra, when $\mathbb{K}$ varies over all imaginary-quadratic number fields.

2 An identity in weight 8

The *Hermitian half-space* $\mathbb{H}_2$ and the *Siegel half-space* $\mathbb{S}_2$ of degree 2 are given by

$$\mathbb{H}_2 := \left\{ Z \in \mathbb{C}^{2\times 2};\ \tfrac{1}{2i}(Z - \overline{Z}^{tr}) > 0 \right\} \supset \mathbb{S}_2 := \left\{ Z \in \mathbb{H}_2;\ Z = Z^{tr} \right\}.$$

Throughout the paper we let $\mathbb{K}$ be an imaginary-quadratic number field of discriminant $\Delta = \Delta_{\mathbb{K}}$ with ring of integers $\mathcal{O}_{\mathbb{K}}$ and inverse different $\mathcal{O}_{\mathbb{K}}^{\sharp} = \mathcal{O}_{\mathbb{K}}\big/\sqrt{\Delta_{\mathbb{K}}}$. If D is a fundamental discriminant, let χ_D denote the associated Dirichlet character; in particular, $\chi_{\mathbb{K}} = \chi_\Delta$. The *Hermitian modular group* of degree 2 is

$$\Gamma_2^{(\mathbb{K})} := \left\{ M \in \mathcal{O}_{\mathbb{K}}^{4\times 4};\ \overline{M}^{tr} J M = J, \det M = 1 \right\},\ J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix},\ I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then

$$\Gamma_2 := \Gamma_2^{(\mathbb{K})} \cap \mathbb{R}^{4\times 4}$$

is the *Siegel modular group* of degree 2. The space $\mathcal{M}_k(\Gamma_2^{(\mathbb{K})})$ of Hermitian modular forms of weight k consists of all holomorphic functions $F : \mathbb{H}_2 \to \mathbb{C}$ satisfying

$$F\big((AZ+B)(CZ+D)^{-1}\big) = \det(CZ+D)^k F(Z) \ \text{ for all } \ \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_2^{(\mathbb{K})}.$$

Any such F has a Fourier expansion of the form

$$F(Z)=\sum_{T\in\Lambda_2,\,T\geqslant 0}\alpha_F(T)e^{2\pi i\,\mathrm{trace}\,(TZ)},$$

where

$$\Lambda_2=\left\{T=\begin{pmatrix} n & t\\ \overline{t} & m\end{pmatrix};\ m,n\in\mathbb{N}_0,\ t\in\mathcal{O}_{\mathbb{K}}^{\sharp}\right\}.$$

If $\varepsilon(T):=\max\{\ell\in\mathbb{N};\ \frac{1}{\ell}T\in\Lambda_2\}$ for $T\neq 0$, we can define the Hermitian Eisenstein series of even weight $k\geqslant 4$ due to [16] and [10] as a Maaß lift via

$$(1)\qquad E_k^{(\mathbb{K})}(Z)=1+\sum_{0\neq T\in\Lambda_2,\,T\geqslant 0}\ \sum_{d|\varepsilon(T)}d^{k-1}\alpha_k^*(|\Delta|\det T/d^2)e^{2\pi i\,\mathrm{trace}\,(TZ)},\quad Z\in\mathbb{H}_2,$$

where $\alpha_k^*=\alpha_{k,\Delta}^*$ is given by

$$(2)\qquad \alpha_k^*(\ell)=\begin{cases}0, & \text{if } \ell\neq 0,\ a_\Delta(\ell)=0,\\ -2k/B_k, & \text{if } \ell=0,\\ r_{k,\Delta}\sum_{t|\ell}\varepsilon_{t,\ell}^{(\Delta)}(\ell/t)^{k-2}, & \text{if } \ell>0,\ a_\Delta(\ell)\neq 0,\end{cases}$$

where

$$(3)\qquad \begin{aligned} r_{k,\Delta} &= \frac{-4k(k-1)}{B_kB_{k-1,\chi}}>0,\\ \varepsilon_{t,\ell}^{(\Delta)} &= \frac{1}{a_\Delta(\ell)}\sum_{\substack{D_1D_2=\Delta\\ D_j\ \text{fund. discr.}}}\chi_{D_1}(t)\chi_{D_2}(-\ell/t),\end{aligned}$$

$$a_\Delta(\ell)=\sharp\{u:\mathcal{O}_{\mathbb{K}}^{\sharp}/\mathcal{O}_{\mathbb{K}};\ \Delta u\overline{u}\equiv\ell \bmod \Delta\}=\prod_{j=1}^{r}(1+\chi_j(-\ell)),$$

if $\Delta=\Delta_1\cdot\ldots\cdot\Delta_r$ is the decomposition into prime discriminants and $\chi_j=\chi_{\Delta_j}$. If $\ell\in\mathbb{N}$ and $a_\Delta(\ell)>0$, then any $t\mid\ell$ satisfies

$$(4)\qquad \begin{aligned}\varepsilon_{t,\ell}^{(\Delta)} &= \prod_{j=1}^{r}\frac{\chi_j(t)+\chi_j(-\ell/t)}{1+\chi_j(-\ell)}\\ &= \prod_{j:\gcd(t,\Delta_j)=1}\chi_j(t)\cdot\prod_{j:\gcd(t,\Delta_j)>1}\chi_j(-\ell/t).\end{aligned}$$

If $k>4$ is even, we have the absolutely convergent series

$$E_k^{(\mathbb{K})}(Z)=\sum_{\left(\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right):\left(\begin{smallmatrix}* & *\\ 0 & *\end{smallmatrix}\right)\backslash\Gamma_2^{(\mathbb{K})}}\det(CZ+D)^{-k},\quad Z\in\mathbb{H}_2.$$

We derive a first result on the growth and the arithmetic of the Fourier coefficients depending on $\mathbb{K}$.

Theorem 1. *Let $\mathbb{K}$ be an imaginary-quadratic number field and let $k \geqslant 4$ be even. Then*

a) $\quad \varepsilon_{t,\ell}^{(\Delta)} = \chi_{D_t'}(t)\chi_{D_t}(-\ell/t)$

holds for all $t|\ell$, $\ell \in \mathbb{N}$, $a_\Delta(\ell) > 0$, where D_t, D_t' are fundamental discriminants satisfying $D_t D_t' = \Delta$, $|D_t| = \gcd(t^\infty, \Delta)$.

b) $\quad 0 < r_{k,\Delta}(2 - \zeta(k-2))\ell^{k-2} \leqslant \alpha_{k,\Delta}^*(\ell) \leqslant r_{k,\Delta}\zeta(k-2)\ell^{k-2}$

holds for all $\ell \in \mathbb{N}$ with $a_\Delta(\ell) > 0$.
c) One has

$$\begin{aligned} 0 &< \frac{(2\pi)^{2k-1}}{\zeta(k-1)\zeta(k)(k-2)!(k-1)!} \cdot \frac{1}{|\Delta|^{k-3/2}} \\ &\leqslant r_{k,\Delta} \leqslant \frac{(2\pi)^{2k-1}}{(2-\zeta(k-1))\zeta(k)(k-2)!(k-1)!} \cdot \frac{1}{|\Delta|^{k-3/2}}. \end{aligned}$$

d) If $\ell_1, \ell_2 \in \mathbb{N}$ are coprime with $a_\Delta(\ell_j) > 0$ and $\gcd(\ell_1\ell_2, \Delta) = 1$, then

$$\alpha_{k,\Delta}^*(\ell_1) \cdot \alpha_{k,\Delta}^*(\ell_2) = r_{k,\Delta} \cdot \alpha_{k,\Delta}^*(\ell_1\ell_2).$$

Proof. We observe that

$$\operatorname{sgn}(B_k B_{k-1,\chi}) = \chi_{\mathbb{K}}(-1) = -1,$$

$$|B_k| = \frac{2k!\zeta(k)}{(2\pi)^k}, \tag{5}$$

$$\frac{2(k-1)!|\Delta|^{k-3/2}}{(2\pi)^{k-1}}(2-\zeta(k-1)) \leqslant |B_{k-1,\chi}| \leqslant \frac{2(k-1)!|\Delta|^{k-3/2}}{(2\pi)^{k-1}}\zeta(k-1), \tag{6}$$

$$\varepsilon_{t,\ell}^{(\Delta)} = \chi_{\mathbb{K}}(t), \quad \text{if} \quad \gcd(t,\Delta) = 1.$$

Then the claim follows from (2), (3) and (4). □

Inserting estimates for the Riemann zeta function we get

$$\tag{7}\begin{aligned}\frac{8\,792}{\sqrt{|\Delta|}} &\leqslant \alpha_4^*(|\Delta|) \leqslant \frac{61\,362}{\sqrt{|\Delta|}},\\ \frac{181\,995}{\sqrt{|\Delta|}} &\leqslant \alpha_6^*(|\Delta|) \leqslant \frac{231\,109}{\sqrt{|\Delta|}},\\ \frac{251\,164}{\sqrt{|\Delta|}} &\leqslant \alpha_8^*(|\Delta|) \leqslant \frac{264\,410}{\sqrt{|\Delta|}},\\ \frac{99\,324}{\sqrt{|\Delta|}} &\leqslant \alpha_{10}^*(|\Delta|) \leqslant \frac{100\,541}{\sqrt{|\Delta|}},\\ \frac{15\,720}{\sqrt{|\Delta|}} &\leqslant \alpha_{12}^*(|\Delta|) \leqslant \frac{15\,768}{\sqrt{|\Delta|}}.\end{aligned}$$

Corollary 1. *Let $\mathbb{K}$ be an imaginary-quadratic number field. Then*

$$E_4^{(\mathbb{K})^2} = E_8^{(\mathbb{K})}$$

holds if and only if $\mathbb{K} = \mathbb{Q}(\sqrt{-3})$.

Proof. If $\mathbb{K} = \mathbb{Q}(\sqrt{-3})$, then the identity follows from [4], Theorem 6. Suppose $|\Delta| \geqslant 4$. The Fourier coefficient of I in $E_4^{(\mathbb{K})^2} - E_8^{(\mathbb{K})}$ is

$$2\alpha_4^*(|\Delta|) + 2\alpha_4^*(0)^2 - \alpha_8^*(|\Delta|) \geqslant \frac{17\,584}{\sqrt{|\Delta|}} + 115\,200 - \frac{264\,410}{\sqrt{|\Delta|}} > 0,$$

according to (7), whenever $|\Delta| \geqslant 5$. For $\Delta = -4$ a direct computation shows that this Fourier coefficient is nonzero. □

Clearly the restriction of $E_4^{(\mathbb{K})^2} - E_8^{(\mathbb{K})}$ to $\mathbb{S}_2$ vanishes because $\dim \mathcal{M}_8(\Gamma_2) = 1$. If $\mathbb{K} = \mathbb{Q}(\sqrt{-1})$ then [4], Theorem 10, yields

$$\tag{8} E_4^{(\mathbb{K})^2} - E_8^{(\mathbb{K})} = c\,\phi_4^2 \quad \text{for} \quad c = 230\,400/61.$$

Corollary 2. *Let $\mathbb{K} = \mathbb{Q}(\sqrt{-1})$. Then the graded ring of symmetric Hermitian modular forms with respect to the maximal discrete extension of $\Gamma_2^{(\mathbb{K})}$ is the polynomial ring in*

$$E_k^{(\mathbb{K})}, \quad k = 4, 6, 8, 10, 12.$$

Proof. [4], Corollary 9 and (8). □

Let $e_{k,m}^{(\mathbb{K})} : \mathbb{H}_1 \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ denote the m-th Fourier-Jacobi coefficient of $E_k^{(\mathbb{K})}$ belonging to $J_{k,m}(\mathcal{O}_\mathbb{K})$, the space of Hermitian Jacobi forms of weight k and index m (cf. [11]). Note that the first Fourier-Jacobi coefficient of $E_4^{(\mathbb{K})^2} - E_8^{(\mathbb{K})}$ vanishes on the submanifold

$\{(\tau,z,z);\ \tau\in\mathbb{H}_1,\ z\in\mathbb{C}\}$. Let $\mathcal{M}_k(\Gamma_1)$ stand for the space of elliptic modular forms of weight k. Then the result of Eichler-Zagier [6], Theorem 3.5, yields

Corollary 3. *Let $\mathbb{K}$ be an imaginary-quadratic number field, $\mathbb{K}\neq\mathbb{Q}(\sqrt{-3})$. If $k\geqslant 4$ is even, the mapping*

$$\mathcal{M}_{k-4}(\Gamma_1)\times\mathcal{M}_{k-6}(\Gamma_1)\times\mathcal{M}_{k-8}(\Gamma_1)\to J_{k,1}(\mathcal{O}_\mathbb{K}),$$
$$(f,g,h)\mapsto f e_{4,1}^{(\mathbb{K})}+g\,e_{6,1}^{(\mathbb{K})}+h\,e_{8,1}^{(\mathbb{K})}$$

is an injective homomophism of the vector spaces, which turns out to be an isomorphism for $\mathbb{K}=\mathbb{Q}(\sqrt{-1})$.

Proof. Note that the dimensions on both sides are equal to $\left[\frac{k}{4}\right]$ due to [4], Theorem 3, whenever $\mathbb{K}=\mathbb{Q}(\sqrt{-1})$. □

We give a precise description of $e_{k,1}^{(\mathbb{K})}$.

Lemma 1. *Let $\mathbb{K}$ be an imaginary-quadratic number field and let $k\geqslant 4$ be even. Then the first Fourier-Jacobi coefficient of $E_k^{(\mathbb{K})}$ is given by*

$$e_{k,1}^{(\mathbb{K})}(\tau,z,w)$$
$$=\frac{1}{2}\sum_{\substack{c,d\in\mathbb{Z}\\ \gcd(c,d)=1}}\ \sum_{\lambda\in\mathcal{O}_\mathbb{K}}(c\tau+d)^{-k}\exp\left(2\pi i\big[(a\tau+b)\lambda\overline{\lambda}-czw+(z\lambda+w\overline{\lambda})\big]/(c\tau+d)\right).$$

Proof. Proceed in the same way as Eichler/Zagier [6] in § 6. One knows that $E_k^{(\mathbb{K})}$ is an eigenform under all Hecke operators $\mathcal{T}_2(p)$ for all inert primes p from [16]. On the other hand the Jacobi-Eisenstein series is an eigenform under

$$\mathcal{T}_J(p)=\Gamma_J^{(\mathbb{K})}\operatorname{diag}(1,p,p^2,p)\Gamma_J^{(\mathbb{K})},$$
$$\Gamma_J^{(\mathbb{K})}=\left\{\left(\begin{smallmatrix}*&*&*&*\\0&0&0&1\end{smallmatrix}\right)\in\Gamma_2^{(\mathbb{K})}\right\},\quad p\in\mathbb{P}\ \text{inert}.$$

As in both cases the constant Fourier coefficient is 1, the claim follows. □

Remark 1. a) If E_k denotes the normalized Eisenstein series of weight k for some group Γ, then the identity $E_4^2=E_8$ is well-known for elliptic modular forms and Siegel modular forms of degree 2 (cf. [19], [7]). But it also holds for modular forms of degree 2 with respect to the Hurwitz order (cf. [15], p. 89) as well as the integral Cayley numbers (cf. [5]), i.e. for $O(2,6)$ and $O(2,10)$. Hence this identity is a hint at the influence of the arithmetic of the attached number field on the modular forms.
b) Note that $E_k^{(\mathbb{K})}$ is a modular form with respect to the maximal discrete extension of $\Gamma_2^{(\mathbb{K})}$ (cf. [17], [22]).
c) It follows from (4) that the Fourier coefficients $\varepsilon_{t,\ell}^{(\Delta)}$ are also multiplicative in Δ, i.e.

$$\varepsilon_{t,\ell}^{(\Delta)}=\varepsilon_{t,\ell}^{(\Delta_1)}\cdot\ldots\cdot\varepsilon_{t,\ell}^{(\Delta_r)}.$$

d) Due to Corollary 3 the dimension of the Maaß space in $\mathcal{M}_k(\Gamma_2^{(\mathbb{K})})$ is $\geqslant \left[\frac{k}{4}\right]$ for $\mathbb{K} \neq \mathbb{Q}(\sqrt{-3})$ and equal to $\left[\frac{k+2}{6}\right]$ for $\mathbb{K} = \mathbb{Q}(\sqrt{-3})$ (cf. [4]), if $k \in \mathbb{N}$ is even.
e) If $k > 4$ is even we can improve the estimate from [16] slightly for all $T \in \Lambda_2$, $T > 0$:

$$\frac{(2\pi)^{2k-1}}{(k-2)!(k-1)!} \cdot \frac{2-\zeta(k-2)}{\zeta(k-1)\zeta(k)} \cdot \frac{1}{\sqrt{|\Delta|}} (\det T)^{k-2} \leqslant \alpha_k(T)$$
$$\leqslant \frac{(2\pi)^{2k-1}}{(k-2)!(k-1)!} \cdot \frac{\zeta(k-3)\zeta(k-2)}{(2-\zeta(k-1))\zeta(k)} \cdot \frac{1}{\sqrt{|\Delta|}} (\det T)^{k-2}.$$

3 Algebraic independence

It is well-known that there are exactly 5 algebraically independent Hermitian modular forms. In this section we explicitly determine algebraically independent Eisenstein series.

We define the Siegel Eisenstein series S_k of degree 2 for even $k \geqslant 4$

$$S_k(Z) = \sum_{\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) : \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \backslash \Gamma_2} \det(CZ+D)^{-k}, \quad Z \in \mathbb{S}_2,$$

and denote its Fourier coefficients by $\gamma_k(R)$. Clearly $E_k^{(\mathbb{K})}|_{\mathbb{S}_2} = S_k$ holds for $k = 4, 6, 8$. The following Fourier coefficients of S_k were computed by Igusa [12] and are given by

f	$\gamma_f \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$	$\gamma_f \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$	$\gamma_f \left(\begin{smallmatrix} 1 & 1/2 \\ 1/2 & 1 \end{smallmatrix}\right)$
S_4	240	30 240	13 440
S_6	-504	166 320	44 352
S_4S_6	-264	-45 360	57 792
X_{12}	65 520	402 585 120	39 957 120

$X_{12} := 441 S_4^3 + 250 S_6^2$.

Lemma 2. *Let $\mathbb{K}$ be an imaginary quadratic number field. Then*

$$F_{10}^{(\mathbb{K})} := E_{10}^{(\mathbb{K})} - E_4^{(\mathbb{K})} E_6^{(\mathbb{K})}, \quad F_{12}^{(\mathbb{K})} := E_{12}^{(\mathbb{K})} - \frac{441}{691} E_4^{(\mathbb{K})^3} - \frac{250}{691} E_6^{(\mathbb{K})^2}$$

are Hermitian cusp forms of weight 10 *resp.* 12, *whose restrictions to* $\mathbb{S}_2$ *do not vanish identically.*

Proof. If $F = F_{10}^{(\mathbb{K})}, F_{12}^{(\mathbb{K})}$, then (1) - (3) show that $\alpha_F(T) = 0$ for all $T \in \Lambda_2$, $\det T = 0$.

Hence F is a cusp form. The Fourier coefficients $\beta_F(R)$ of $F|_{\mathbb{S}_2}$ are given by

$$\beta_F(R) = \sum_{\substack{T\in\Lambda_2,\, T\geqslant 0\\ T+\overline{T}=2R}} \alpha_f(T).$$

Note that $E_k^{(\mathbb{K})}|_{\mathbb{S}_2} = S_k$ for $k = 4, 6, 8$. If $k = 10$, then Theorem 1 and the above table yield $\beta_F(I) > 0$.
If $k = 12$, then for $\Delta \neq -4$

$$\begin{aligned}\beta_F(I) &= \alpha^*_{12}(\Delta) + 2 \sum_{1\leqslant j<\sqrt{|\Delta|}} \alpha^*_{12}(|\Delta| - j^2) - \frac{402\,585\,120}{691}\\ &\leqslant 15\,768 \left(\frac{1}{\sqrt{|\Delta|}} + 2\right) - \frac{402\,585\,120}{691} < 0\end{aligned} \tag{9}$$

by means of Theorem 1. If $\Delta = -4$ then

$$\beta_F(I) = -\frac{20\,026\,621\,440\,000}{34\,910\,011} < 0.$$

□

A simple consequence is

Theorem 2. *Let $\mathbb{K}$ be an imaginary-quadratic number field.*
a) The graded ring of Siegel modular forms of even weight is generated by

$$E_k^{(\mathbb{K})}|_{\mathbb{S}_2},\ k = 4, 6, 10, 12.$$

b) If $\mathbb{K} \neq \mathbb{Q}(\sqrt{-3})$ the Eisenstein series

$$E_k^{(\mathbb{K})},\ k = 4, 6, 8, 10, 12$$

are algebraically independent.

Proof. a) Use Lemma 2.
b) Apply a) as well as Corollary 1. □

If $\mathbb{K} = \mathbb{Q}(\sqrt{-3})$, we already know the graded ring of Hermitian modular forms (cf. [4], Theorem 6).

Corollary 4. *If $\mathbb{K} = \mathbb{Q}(\sqrt{-3})$ the graded ring of symmetric Hermitian modular forms of even weight with respect to $\Gamma_2^{(\mathbb{K})}$ is the polynomial ring in*

$$E_k^{(\mathbb{K})},\ k = 4, 6, 10, 12, 18.$$

Proof. Use [4], Theorem 6, and show that

$$E_{18}^{(\mathbb{K})},\ E_{12}^{(\mathbb{K})}\cdot E_6^{(\mathbb{K})},\ E_{10}^{(\mathbb{K})}\cdot {E_4^{(\mathbb{K})}}^2,\ {E_6^{(\mathbb{K})}}^3,\ E_6^{(\mathbb{K})}\cdot {E_4^{(\mathbb{K})}}^3$$

are linearly independent by calculating a few Fourier coefficients using (1) - (4). □

Remark 2. a) It follows from the results of [3] that there is a non-trivial cusp form $f_4^{(\mathbb{K})}$ of weight 4 for all discriminants except $\Delta_{\mathbb{K}} = -3, -4, -7, -8, -11, -15, -20, -23$. As its restriction to the Siegel half-space vanishes identically, one may replace $E_8^{(\mathbb{K})}$ by $f_4^{(\mathbb{K})}$ in these cases in Theorem 2 b).
b) Using Theorem 2 resp. Corollary 4 resp. part a) we can construct a non-trivial skew-symmetric Hermitian modular form of weight 44 resp. 54 resp. 40 by an application of a suitable differential operator (cf. [1]).

4 A Siegel cusp form

We consider

$$(10)\qquad \begin{aligned} G_k^{(\mathbb{K})}(Z) &:= E_k^{(\mathbb{K})}(Z) - S_k(Z), \quad Z \in \mathbb{S}_2,\\ &= \sum_{\substack{\left(\begin{smallmatrix} A & B\\ C & D\end{smallmatrix}\right) : \left(\begin{smallmatrix} * & *\\ 0 & *\end{smallmatrix}\right)\backslash \Gamma_2^{(\mathbb{K})}\\ C^\sharp D \notin \mathbb{R}^{2\times 2}}} \det(CZ+D)^{-k}, \end{aligned}$$

where $\left(\begin{smallmatrix} c_1 & c_2\\ c_3 & c_4\end{smallmatrix}\right)^\sharp = \left(\begin{smallmatrix} c_4 & -c_2\\ -c_3 & c_1\end{smallmatrix}\right)$. For $\mathbb{K} = \mathbb{Q}(\sqrt{-1})$ this modular form was introduced by Nagaoka and Nakamura [20].

Theorem 3. *Let $\mathbb{K}$ bei an imaginary-quadratic number field. If $k \geqslant 10$ is even, then $G_k^{(\mathbb{K})}$ is a Siegel cusp form of degree 2 and weight k in the Maaß space.*
a) One has

$$(11)\qquad \lim_{|\Delta_{\mathbb{K}}|\to\infty} G_k^{(\mathbb{K})}(Z) = 0 \ \text{ for all } \ Z \in \mathbb{S}_2.$$

b) $G_k^{(\mathbb{K})} \not\equiv 0$ *holds whenever* $k \geqslant \frac{10}{3}|\Delta_{\mathbb{K}}| + 1$.

Proof. $G_k^{(\mathbb{K})}$ is a cusp form, as all its Fourier-coefficients $\beta_k(R)$ with $\det R = 0$ vanish due to (1) - (4). It belongs to the Maaß space by virtue of [16] and [10].
a) Because $\dim \mathcal{M}_k(\Gamma_2) \leqslant 1$ for $k < 10$ (cf. [7]), we have $G_k^{(\mathbb{K})} \equiv 0$ for $k = 4, 6, 8$. Let $k \geqslant 10$ and

$$S_1 = \begin{pmatrix} 0 & 0 & P\\ 0 & -S & 0\\ P & 0 & 0\end{pmatrix},\ P = \begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix},\ S = \begin{pmatrix} 2 & 0\\ 0 & |\Delta|/2\end{pmatrix},\ \text{resp.}\ \begin{pmatrix} 2 & 1\\ 1 & (|\Delta|+1)/2\end{pmatrix},$$

if Δ is even resp. odd. Then the explicit isomorphism in [17] yields

$$(12)\qquad G_k^{(\mathbb{K})}(Z) = \sum_{\substack{h\in\mathbb{Z}^6,\, h_4\geqslant 1\\ h^{tr}S_1h=0,\, \gcd(S_1h)=1}} \left(-h_1 \det Z + \operatorname{trace}\left(\begin{pmatrix} h_2 & g\\ \overline{g} & h_5\end{pmatrix}\cdot Z\right) + h_6\right)^{-k},$$

where $g = h_3 + h_4\sqrt{\Delta}/2$ resp. $g = h_3 + h_4(1+\sqrt{\Delta})/2$. By virtue of [15], V.2.5, it suffices to show that the series in (12) over the absolute values at $Z = iI$ tends to 0, as $|\Delta| \to \infty$. Hence we consider

$$\begin{aligned} I_\Delta &:= \sum_h \left|h_1 + h_6 + i(h_2+h_5)\right|^{-k}\\ &= \sum_h \left(h_1^2 + h_6^2 + h_2^2 + h_5^2 + (h_3\, h_4)S\begin{pmatrix} h_3\\ h_4\end{pmatrix}\right)^{-k/2}\end{aligned}$$

in view of $h^{tr}S_1h = 0$. As $h_4 \geqslant 1$ we get

$$I_\Delta \leqslant \sum_h \left(\frac{|\Delta|-3}{2} + \frac{1}{2}h^{tr}h\right)^{-k/2} \leqslant (|\Delta|-3)^{-k/4}\cdot \sum_h (h^{tr}h)^{-k/4}.$$

If $k \geqslant 14$ we can use the Epstein zeta function for $I^{(6)}$ in order to obtain $\lim_{|\Delta|\to\infty} I_\Delta = 0$. For $k = 10$ and $k = 12$ apply

$$a + b \geqslant \sqrt[4]{a}\cdot\sqrt[4]{b^3} \ \text{ for } \ a, b > 0$$

and proceed in the same way.
b) Let Δ be fixed. Then the Fourier coefficient $\beta_k(I)$ of $G_k^{(\mathbb{K})}$ is given by

$$\beta_k(I) = \alpha_k^*(\Delta) + 2\sum_{1\leqslant j\leqslant\sqrt{\Delta}} \alpha_k^*(\Delta - j^2) - \gamma_k(I).$$

Due to Maaß [18]

$$0 < \gamma_k(I) = \frac{-4kB_{k-1,\chi_{-4}}}{B_k B_{2k-2}}$$

holds. Using Corollary 1 this leads to

$$\begin{aligned} \beta_k(I) &\geqslant r_{k,\Delta}\left((2-\zeta(k-2))\left(|\Delta|^{k-2} + 2\sum_{1\leqslant j\leqslant\sqrt{|\Delta|}} (|\Delta|-j^2)^{k-2}\right) - \frac{B_{k-1,\chi_{-4}}B_{k-1,\chi}}{(k-1)B_{2k-2}}\right)\\ &\geqslant r_{k,\Delta}|\Delta|^{k-3/2}\left(\frac{2-\zeta(k-2)}{\sqrt{|\Delta|}} - \frac{2^{2k-2}(k-1)!^2\zeta(k-1)^2}{(k-1)\zeta(2k-2)(2k-2)!}\right)\end{aligned}$$

for $\Delta \neq -4$, if we use (5) and (6). Then Stirling's formula leads to

$$\beta_k(I) \geqslant r_{k,\Delta}|\Delta|^{k-3/2}\left(\frac{2-\zeta(k-2)}{\sqrt{|\Delta|}} - \frac{e^{1/6(k-1)}\zeta(k-1)^2}{\zeta(2k-2)}\sqrt{\frac{\pi}{k-1}}\right)$$
$$\geqslant r_{k,\Delta}|\Delta|^{k-3/2}\left(\frac{2-\zeta(8)}{\sqrt{|\Delta|}} - e^{1/54}\zeta(9)^2\sqrt{\frac{\pi}{k-1}}\right),$$

as $k \geqslant 10$. The expression in the bracket is positive because

$$k-1 \geqslant \tfrac{10}{3}|\Delta| > \pi\left(\frac{e^{1/54}\zeta(9)^2}{2-\zeta(8)}\right)^2|\Delta|.$$

If $\Delta = -4$ we compute $\beta_k(R)$, $R = \left(\begin{smallmatrix} 1 & 1/2 \\ 1/2 & 1\end{smallmatrix}\right)$, and proceed in the same way (cf. [20]). □

Use the description of $\gamma_k(R)$ by means of Cohen's function in [6], p. 80. A comparison of the Fourier coefficients and the Hecke bound for the Fourier coefficients of cusp forms yield the following asymptotic.

Corollary 5. *If $k \geqslant 4$ is even and $N \in \mathbb{N}$, $N \equiv 0, 3 \bmod 4$ one has*

$$H(k-1,N) \sim \sum_{\substack{|j|\leqslant\sqrt{|\Delta|N} \\ j\equiv\Delta N \bmod 2}} \alpha^*_{k,\Delta}\big((|\Delta|N - j^2)/4\big)$$

as $N \to \infty$ for any imaginary-quadratic number field $\mathbb{K}$.

Remark 3. a) We know that $G_k^{(\mathbb{K})} \equiv 0$ for $k = 4, 6, 8$. Hence we get equality for $k = 4, 6, 8$ in Corollary 5. We conjecture that $G_k^{(\mathbb{K})} \not\equiv 0$ for any even $k \geqslant 10$ and any imaginary-quadratic number field $\mathbb{K}$. This has been verified for $|\Delta_{\mathbb{K}}| \leqslant 500$ by the authors. The Fourier coefficients are not always positive as in the proof of Theorem 3 (cf. [9]).
b) The paramodular group of level t can be embedded into $\Gamma_2^{(\mathbb{K})}$, whenever t is the norm of an element in $\mathcal{O}_{\mathbb{K}}$ (cf. [13]). Hence one can construct paramodular cusp forms in the Maaß space in the same way.

5 The Maaß Spezialschar

At first we characterize the vanishing of $G_k^{(\mathbb{K})}$. Recall (e.g. [21], Theorem 3.14) that for even $k \in \mathbb{N}$ the *Shimura lifts* are maps

$$Sh_{k,t-1/2} : \mathcal{S}_{k-1/2}(\Gamma_0(4)) \to \mathcal{S}_{2k-2}(SL_2(\mathbb{Z}))$$
$$f(\tau) = \sum_{n=1}^{\infty} a(n) e^{2\pi i n\tau} \mapsto \sum_{n=1}^{\infty} b_t(n) e^{2\pi i n\tau},$$

for squarefree $t \in \mathbb{N}$, where the coefficients $b_t(n)$ are given by

$$\sum_{n=1}^{\infty} b_t(n) n^{-s} = \sum_{n=1}^{\infty} \left(\frac{t}{n}\right) n^{k-s-3/2} \cdot \sum_{n=1}^{\infty} a(tn^2) n^{-s}.$$

Let $\mathcal{S}^+_{k-N_2}(\Gamma_0(4))$ be Kohnen's plus space, i.e.

$$a(n) = 0 \ \text{ for } \ n \equiv 1, 2 \bmod 4.$$

Lemma 3. *Let $k \geqslant 4$ be even and $\mathbb{K} = \mathbb{Q}(\sqrt{-t})$, $t \in \mathbb{N}$ squarefree. Then the following are equivalent:*

(i) The restriction of the Hermitian Eisenstein series $E_k^{(\mathbb{K})}|_{\mathbb{S}_2}$ is equal to the Siegel Eisenstein series S_k.

(ii) The t-Shimura lift $Sh_{t,k-1/2} : \mathcal{S}^+_{k-1/2}(\Gamma_0(4)) \to \mathcal{S}_{2k-2}(SL_2(\mathbb{Z}))$ is identically zero.

Proof. As $E_k^{(\mathbb{K})}|_{\mathbb{S}_2}$ is the Maaß lift of $e_k^{(\mathbb{K})}(\tau, z, z)$, condition (i) holds if and only if $e_k^{(\mathbb{K})}(\tau, z, z)$ is equal to the classical Jacobi-Eisenstein series in [6]. Due to Lemma 1 one has

$$e_k^{(\mathbb{K})}(\tau, z, z) = \sum_{\lambda \in \mathcal{O}_{\mathbb{K}}} \sum_{M : \Gamma_\infty \backslash \Gamma_1} e^{2\pi i(\tau\lambda\overline{\lambda} + z(\lambda+\overline{\lambda}))}|_{k,1} M,$$

if we use the definition of the slash operator from [6]. For each fixed λ the series

$$\sum_{M : \Gamma_\infty \backslash \Gamma_1} e^{2\pi i(\tau\lambda\overline{\lambda} + z(\lambda+\overline{\lambda}))}|_{k,1} M = P_{k,\lambda\overline{\lambda},\lambda+\overline{\lambda}}(\tau, z)$$

is the Jacobi-Poincaré series (cf. [23]). Its Petersson inner product with a cusp form ϕ is, up to a factor depending on k as well as $(\lambda - \overline{\lambda})^{6-4k}$, equal to the Fourier coefficient of the term $e^{2\pi i(\tau\lambda\overline{\lambda} + z(\lambda+\overline{\lambda}))}$. Thus (i) holds if and only if

$$\sum_{0 \neq \lambda \in \mathcal{O}_{\mathbb{K}}} (\lambda - \overline{\lambda})^{6-4k} c(\lambda\overline{\lambda}, \lambda + \overline{\lambda}) = 0$$

holds for every Jacobi cusp form

$$\phi(\tau, z) = \sum_{n,r} c(n,r) e^{2\pi i(n\tau + rz)} \in \mathcal{J}_{k,1}. \tag{13}$$

We want to rephrase this in terms of half-integral weight modular forms for $\Gamma_0(4)$ as in [6], § 5. Given (13) we attach the modular form

$$F(\tau) = \sum_{\substack{N\in\mathbb{N} \\ N=4n-r^2}} c(n,r) e^{2\pi i N\tau} \in \mathcal{M}_{k-1/2}(\Gamma_0(4)),$$

satisfying Kohnens's plus condition. This bijection respects the Petersson inner products up to a trivial factor. The half-integral weight modular form attached to $P_{k,\lambda\overline{\lambda},\lambda+\overline{\lambda}}$ is therefore the projection to the Kohnen plus space of the Poincaré series of weight $k-1/2$ and index $-(\lambda-\overline{\lambda})^2$. As λ runs through $\mathcal{O}_{\mathbb{K}}$ these values run through $|\Delta|n^2$, $n \in \mathbb{N}_0$. Hence (i) holds if and only if

$$\sum_{n=1}^{\infty} c(|\Delta|n^2) n^{3-2k} = 0$$

for every cusp form $F(\tau) = \sum_{N\equiv 0,3 \bmod 4} c(N) e^{2\pi i N\tau} \in S_{k-1/2}(\Gamma_0(4))$. If such an F is a Hecke eigenform, then its Shimura lift $Sh_{t,k-1/2}(F)$ ist either a Hecke eigenform with the same eigenvalues or identically 0. In the first case the Euler product of its L-function implies that

$$\sum_{n=1}^{\infty} b_t(n) n^{3-2k} \neq 0$$

and thus

$$\sum_{n=1}^{\infty} a(tn^2) n^{3-2k} \neq 0.$$

Therefore (i) holds if and only if all the Hecke eigenforms in $\mathcal{S}^+_{k-1/2}(\Gamma_0(4))$ map to 0 under $Sh_{t,k-1/2}$. □

This leads to our final result

Corollary 6. *Let $k \geqslant 10$ be even. Then the modular form*

$$G_k^{(\mathbb{K})} = E_k^{(\mathbb{K})}\big|_{\mathbb{S}_2} - S_k$$

generate the subspace of cusp forms in the Maaß Spezialschar as a module over the Hecke algebra, as $\mathbb{K}$ varies over all imaginary-quadratic number fields.

Proof. Let $\mathcal{A}_{k-1/2}(\Gamma_0(4)) \subseteq \mathcal{S}^+_{k-1/2}(\Gamma_0(4))$ be the subspace generated by the preimages of $E_k^{\mathbb{Q}(\sqrt{t})}\big|_{\mathbb{S}_2} - S_k$ under the Maaß lift as t runs through all squarefree numbers in $\mathbb{N}$. In the proof of Lemma 3 it was shown that a Kohnen plus Hecke eigenform $F \in \mathcal{S}^+_{k-1/2}(\Gamma_0(4))$

is orthogonal to $\mathcal{A}_{k-1/2}(\Gamma_0(4))$ if and only if its t-Shimura lifts $Sh_{t,k-1/2}(F)$ are 0 for all squarefree $t \in \mathbb{N}$. This cannot happen by a theorem of Kohnen [14], which guarantees that some linear combination of Shimura lifts yields a Hecke-equivariant isomorphism

$$\mathcal{S}^+_{k-1/2}(\Gamma_0(4)) \xrightarrow{\sim} \mathcal{S}_{2k-2}(\Gamma_1).$$

It follows that $\mathcal{A}_{k-1/2}(\Gamma_0(4))$ generates $\mathcal{S}^+_{k-1/2}(\Gamma_0(4))$ as a module over the Hecke algebra. Since Maaß lifts respect Hecke operators, we obtain the claim. □

Remark 4. a) Lemma 3 is trivial for $k = 4, 6, 8$ because

$$\mathcal{S}_k(\Gamma_2) = \mathcal{S}_{2k-2}(\Gamma_1) = \{0\}.$$

b) It is an open question whether the modular forms $G_k^{(\mathbb{K})}$ span the space of cusp forms in the Maaß space of even weight $k \geqslant 10$, when $\mathbb{K}$ runs over all imaginary-quadratic number fields.

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