APPENDIX TO A PAPER OF B. WILLIAMS

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ABSTRACT. In this appendix to a paper by B. Williams, we give birational equivalences between the models of the Hilbert modular surfaces for $\mathbb{Q}(\sqrt{29})$ and $\mathbb{Q}(\sqrt{37})$ given there and those previously found by Elkies and Kumar.

1. INTRODUCTION

The main results of the paper [7] give explicit presentations of the ring of Hilbert modular forms for the full ring of integers of $\mathbb{Q}(\sqrt{29})$ and $\mathbb{Q}(\sqrt{37})$. Applying the **Proj** functor to these rings gives the *Baily-Borel compactification* [6, Theorem 2.7.1] of the associated Hilbert modular varieties. However, these schemes are somewhat inconvenient to work with in a computer algebra system, because they are embedded in a highly singular weighted projective space of large dimension. In addition, the cusp singularities, though they admit elegant and meaningful resolutions described in [6, Section 2.6], are not canonical ([4, (1.1), (1.2)]), and therefore interfere with calculations of linear systems on the surfaces.

On the other hand, the paper [2] gives many models of Hilbert modular surfaces as subvarieties of ordinary or weighted projective space with only canonical singularities, and as elliptic surfaces over a rational curve. These are very convenient for such purposes as calculating rational curves and elliptic fibrations on the surface, but they do not represent easily described functors of families of abelian surfaces with real multiplication. Thus it may not be clear, for example, how to describe the Hirzebruch-Zagier divisors [6, Chapter 5] on them.

To bridge the gap, therefore, it is convenient to exhibit an explicit birational equivalence between the Baily-Borel compactifications of [7] and the elliptic surfaces of [2]. We find these in two stages:

(1) By repeatedly projecting away from bad components of the singular locus, we obtain a model of the surface in projective space with only canonical singularities. (Recall from, for example, [4, (1.2)] that canonical singularities of a surface are exactly those with a resolution by rational curves of self-intersection -2 whose intersection graph is a Dynkin diagram of type A, D, E.)

(2) Having found a suitable model, we find rational curves and elliptic fibrations on it until we come across a fibration that matches the description given in [2]. This is an iterative process: given some rational curves, we can find the ADE configurations supported on them and write down the corresponding elliptic fibrations; in turn, an elliptic fibration on a K3 surface of large

Picard rank is likely to have reducible fibres, which are composed of rational curves.

It is not clear how to guarantee success for either of these stages, but in both of the examples discussed here this process could be carried through in Magma [1]. The code supporting the claims of this appendix is available as [3].

2. Finding a model with canonical singularities

We recall [6, Theorem 7.3.3] that the Hilbert modular surfaces for $\mathbb{Q}(\sqrt{29})$ and $\mathbb{Q}(\sqrt{37})$ are birationally equivalent to K3 surfaces. If D is a nef and big divisor on a K3 surface, then by Riemann-Roch we have $D^2 = 2h^0(D) - 4$. So if $S \subset \mathbb{P}^n$ is birational to a K3 surface, we can use $\deg S - 2n$ as a measure of how far we are from finding a canonical model given by a complete linear series.

Our first step for both surfaces will be to eliminate some of the variables to obtain a surface in the weighted projective space $\mathbb{P}(2, 2, 3, 3, 6)$. This space is embedded in $\mathbb{P}^7$ by $\mathcal{O}(6)$, so we obtain a birationally equivalent surface in ordinary projective space. The general way to improve a singularity is to blow up along a component of the singular locus, which in computer algebra can often be interpreted as a map $\mathbb{P}^n \dashrightarrow \mathbb{P}^n \times \mathbb{P}(a_1, \ldots, a_k)$, where the a_i are the degrees of the equations defining the component. In a situation such as ours, however, this would rapidly lead to an unmanageable profusion of variables. Instead we use two basic steps: projection away from (the linear span of) a component of the singular locus and the 2-uple Veronese embedding. The disadvantage of this is that there is no guarantee that a step will actually improve the singularities; projecting away from a point, for example, is locally the same as blowing it up, if there are no lines through the point. Such lines will be contracted to new singularities. In addition, the projection may be a map of degree 2 rather than a birational equivalence to its image. We will start by projecting away from 1-dimensional components of the singular subscheme and then proceed to components of large degree; although the degree of the component of the singular locus of a scheme at a singularity of type A_i, D_i, E_i is i and may be arbitrarily large in the first two cases, each such singularity contributes i to the Picard number, which is at most 20 for a K3 surface in characteristic 0. Thus we expect that a component of the singular locus of degree much greater than 10 is not a canonical singularity (and can verify this expectation using Magma if necessary).

We now give the details of the construction, first for $\mathbb{Q}(\sqrt{29})$ and then for $\mathbb{Q}(\sqrt{37})$. For modular forms, we use the notation of [7]; for coordinates on ordinary projective space $\mathbb{P}^n$, we use $x_0, \ldots, x_n$. It is never necessary to consider more than one projective space at a time, so no confusion will result from this.

2.1. $\mathbb{Q}(\sqrt{29})$. As noted above, we begin by defining a rational map from the Baily-Borel compactification in the weighted projective space $\mathbb{P}(2, 2, 3, 3, 4, 5, 6, 6, 7, 8, 9)$ to a surface in $\mathbb{P}(2, 2, 3, 3, 6)$ by the equations $(E_2 : \phi_2 : \phi_3 : \psi_3 : \phi_6)$. Every other generator of the ring of modular forms can be eliminated using a relation of degree 1 in it (these relations can be chosen in order so as not to reintroduce previously eliminated generators). Making these substitutions introduces two extraneous components to the scheme, supported at $\phi_2 = \phi_3 = 0$ and $E_2 = \phi_2 = \phi_3 = 0$, but these are easily removed. Since one of the equations for the desired component is $E_2\phi_2^2 + 4\phi_2^3 - \phi_3^2 + 4\phi_3\psi_3$ of degree 6, the embedding by $\mathcal{O}(6)$ goes naturally into $\mathbb{P}^6$; the image has degree 15.

The singular subscheme of the image in $\mathbb{P}^6$ includes a highly nonreduced component of degree 45 supported at $(1 : 0 : 0 : 0 : 0 : 0 : 0)$. We project away from this point to obtain a surface of degree 11 in $\mathbb{P}^5$. In turn, the singular subscheme of this surface has a component of degree 21 supported at $(0 : 0 : 0 : 0 : 0 : 1)$; projecting away from this point gives a surface of degree 8 in $\mathbb{P}^4$, which is a complete intersection. It does not have any isolated rational singular points to project from, so it is time to use the Veronese embedding. We consider the map given by the linear system of quadrics that vanish on the two singular lines $x_0 = x_1 = x_2 = 0$ and $x_0 = x_2 = x_3 = 0$ as well as on the isolated singular points $(\pm\frac{\sqrt{-1}}{4} : 0 : 0 : 0 : 1)$, modulo the quadric vanishing on the surface. This gives a map to a surface in $\mathbb{P}^6$, whose singularities are all isolated points.

The surface in $\mathbb{P}^6$ has a bad singularity at $(0 : 0 : 0 : 0 : 0 : 0 : 1)$, but projection from this point is not a birational equivalence from the surface to its image. Instead, we project away from the mild singularities at $(-1 : 0 : 1 : 0 : 0 : 0 : 0)$ and $(0 : 0 : 0 : 1 : 0 : 0 : 0)$, obtaining a second surface in $\mathbb{P}^4$. On this surface we consider the linear system of quadrics vanishing on the reduced subscheme of the singular locus, which consists of two lines defined over $\mathbb{Q}$ and two triples of points defined over cubic fields (in fact the same cubic fields that will appear later as fields of definition of the bad fibres of the elliptic fibration). After taking the quotient by the quadric in the ideal of the surface in $\mathbb{P}^4$, this gives us a map to $\mathbb{P}^5$. The only bad singularity of the image is at $(0 : 0 : 0 : 0 : 0 : 1)$, and projecting away from it gives a surface $S_{29} \subset \mathbb{P}^4$ with four isolated double points and no other singularities. This surface is defined by the equations

$$
\begin{aligned}
-48x_1^2 &+ 68x_1x_2 + 3x_2^2 + 20x_0x_3 + 232x_1x_3 - 42x_2x_3 - 181x_3^2 - 3x_4^2 = \\
320x_0x_1^2 &+ 2496x_1^3 + 880x_0x_1x_2 - 6560x_1^2x_2 - 20x_0x_2^2 + 3628x_1x_2^2 + 114x_2^3 \\
&- 17584x_1^2x_3 + 20124x_1x_2x_3 - 2441x_2^2x_3 + 29244x_1x_3^2 - 13032x_2x_3^2 \\
&- 14117x_3^3 + 20x_0x_4^2 + 156x_1x_4^2 - 114x_2x_4^2 - 195x_3x_4^2 = 0.
\end{aligned} \tag{1}
$$

Being a complete intersection of degrees $2, 3$ in $\mathbb{P}^4$ with only canonical singularities, it is a birational model of a K3 surface. It is easily checked that every step along the way was invertible, so it is birationally equivalent to the original surface.

2.2. $\mathbb{Q}(\sqrt{37})$. This calculation is quite similar to the one just presented. We use the notation of the supplementary material to [7] for modular forms: in terms of the notation in the paper, we have

$$
g_2 = \phi_1\psi_1, \quad g_{3,1} = \phi_1^3, \quad g_{3,2} = \phi_1\psi_2, \quad g_{6,2} = \psi_2\phi_4. \tag{2}
$$

We begin by expressing the other generators in terms of E_2 and these four, which gives a surface in $\mathbb{P}(2, 2, 3, 3, 6)$, and embedding that surface into $\mathbb{P}^6$ (again there is an equation of degree 6). This surface has degree 24 rather than 15 as in the previous example, so we expect the reduction process to take a bit longer.

Now we project away from the worst isolated singularity at $(0 : 0 : 0 : 0 : 0 : 0 : 1)$, and then from the image of the second worst at $(1 : 0 : 0 : 0 : 0 : 0)$. After that the worst isolated singularity is at $(0 : 1 : 0 : 0 : 0)$; projection gives a surface of degree 7 in $\mathbb{P}^3$. We have decreased the invariant $\deg S - 2n$ by 11, but there is still work to do.

We map back to $\mathbb{P}^6$ by the forms of degree 2 vanishing at the three worst singular points $(0 : 0 : 0 : 1), (0 : 0 : 1 : 0), (1 : 0 : 0 : 0)$, obtaining a surface in $\mathbb{P}^6$. This

surface is singular along $x_5 = x_6 = 0$, so we project away from that line. From the resulting surface in $\mathbb{P}^4$, we map by forms of degree 2 vanishing on the reduced subscheme of the singular locus mod those vanishing on the surface to get a surface of degree 6 in $\mathbb{P}^3$. In turn, we map by quadrics vanishing along the singular line $x_2 = x_3 = 0$ and the point $(-1 : 1 : 0 : 0)$ to a surface in $\mathbb{P}^5$, which has a bad singularity at $(-1 : 1 : -1 : 1 : 0 : 0)$. Projecting away from this, we return to $\mathbb{P}^4$, and from there we map by quadrics vanishing on the 1-dimensional components of the singular locus (whose degrees are $2, 1, 1$) mod those vanishing on the surface to obtain a surface S_{37} in $\mathbb{P}^5$. This surface is defined by the equations

$$
\begin{aligned}
&- x_3 x_4 + x_0 x_5 + 2 x_1 x_5 + x_2 x_5 = \\
&- 243 x_1^2 + 243 x_0 x_2 + 162 x_1 x_3 - 324 x_2 x_3 + 3 x_4^2 - 8 x_4 x_5 + 4 x_5^2 = \\
&2187 x_0^2 + 8748 x_0 x_1 - 5184 x_0 x_3 - 6480 x_1 x_3 - 1296 x_2 x_3 \\
&+ 2592 x_3^2 + 40 x_4 x_5 - 76 x_5^2 = 0.
\end{aligned}
\tag{3}
$$

This is a smooth complete intersection of quadrics in $\mathbb{P}^5$, so it is a K3 surface. Again all maps are easily checked to be invertible.

3. Finding rational curves and elliptic fibrations

We now indicate how to find elliptic fibrations on the surfaces defined by (1), (3) that match those of [2, Sections 16, 18]. We do not use a systematic procedure for this; we simply list rational curves, group them into fibres of elliptic fibrations, and hope for the best. The basic fact (going back to Pyatetskiĭ-Shapiro and Shafarevich if not beyond) that allows us to construct elliptic fibrations is the following:

Theorem 3.1. *Let $R_1, \ldots, R_k$ be smooth rational curves on a K3 surface in the configuration of a reducible fibre of a minimal elliptic fibration. Then there is a genus 1 fibration on the surface one of whose fibres is supported on the R_i.*

Indeed, a suitable linear combination of the R_i is effective and nonzero and has self-intersection 0 and no base components. The result then follows from Riemann-Roch. (See, for example, [5, Table 4.1] for the list of configurations.)

Once we know that an elliptic fibration exists, we still have to construct it. This is quite easy on S_{37} (3), because it is smooth: given a scheme-theoretic fibre F, consider a polynomial p of degree d (to be chosen as small as possible) that vanishes on F but not on the whole surface. Let $R = (p = 0) \setminus F$ be the residual divisor; then for any form q of degree d vanishing on R, the residual $(q = 0) \setminus R$ is linearly equivalent to F. So the linear system of such polynomials modulo those vanishing on the surface defines the fibration (we are implicitly using the projective normality of surfaces defined by a complete linear system). On the singular surface S_{29} (1), things could be slightly more difficult if we had to consider fibres with multiple components that meet the singular locus or that are supported there, but we do not so the same method works.

After equations defining a fibration are found, it is a simple matter to write down the fibre over the generic point of $\mathbb{P}^1$ and (in the cases described here where the fibre is of low degree) to find an explicit map to a Weierstrass model. Using Magma we can then find the reducible fibres and check whether they match those of [2]. If not, we instead decompose them into new rational curves, form them into new fibres, and repeat the process.

3.1. $\mathbb{Q}(\sqrt{29})$. We are working on a surface with four singular points of which one is rational. The tangent cone at the singular point meets the surface in a conic C_1, two lines L_1, L_2, where L_1 contains $(0 : 0 : -1 : 0 : 1)$ and L_2 contains $(0 : 0 : 1 : 0 : 1)$, and two conjugate lines defined over $\mathbb{Q}(\sqrt{-1})$. When we project away from the singular point, we obtain a quartic in $\mathbb{P}^3$ with two rational singular points (the images of the rational lines). Pulling back the curves in the tangent cones at these singular points, we obtain two additional lines L_3, L_4 not containing the singular point of S_{29}: for definiteness, let L_3 be the one that contains $(1 : 0 : -1 : 0 : 1)$.

The exceptional divisor above the singular point and the three geometrically irreducible components of the intersection with the tangent cone constitute an I_4 configuration, so we may form an elliptic fibration. It is defined by the equations $(x_2 : x_3)$. In addition to the I_4 fibre, it also has an I_2 fibre consisting of two conics C_2, C_3 that meet in two points. Let C_2 be the one that is not locally solvable at $3, \infty$ and C_3 the one with rational points such as $(-2 : -1 : 2 : -1 : 1)$.

The curves L_3, L_4, C_3 now form an I_3 fibre. The associated fibration also has an I_2 fibre and another I_3, and is defined by $(x_0 + 6x_2/5 : x_1 + x_2/3 - x_3)$. From the I_3 fibre that contains C_1 we obtain a new conic C_4 which is the union of two lines defined over $\mathbb{Q}(\sqrt{-3})$, while the I_2 fibre affords a curve Q_1 of degree 4.

We now find some additional curves by mapping to another model in projective space. To be precise, consider the forms of degree 2 that vanish on C_1, L_1, C_4 mod those vanishing on S_{29}. These give us a map to a surface in $\mathbb{P}^3$, and pulling back the singular points and components of their tangent cones we find three new conics C_5, C_6, C_7 of which C_5 contains $(0 : 1 : 0 : 1 : 1)$, while C_6 is reducible over $\mathbb{Q}(\sqrt{-1})$ and C_7 contains $(-6/5 : 0 : 1 : 0 : 1)$.

The fibration with $C_2 \cup C_3 \cup C_5$ is the one we are looking for. As in [2, Section 14], it has a rational I_4 fibre, a rational I_3 and three defined over the cubic field of discriminant -87, and three I_2 fibres defined over the cubic field of discriminant -116. In addition, there is a rational fibre of type II. It is easy to change coordinates so that the I_4, I_3, II fibres of the two are above the same points of $\mathbb{P}^1$. This only shows that the Jacobian of the fibration we constructed on S_{29} is isomorphic to the fibration of [2], but L_1 is a section of the fibration, so the two are isomorphic and we have found the birational equivalence. The defining equations for this fibration (after changing coordinates) are

$$(x_0 - 3x_1/5 - 7x_3/5 : x_2 + 2x_3).$$

3.2. $\mathbb{Q}(\sqrt{37})$. In this case it is a bit more difficult to get started, because we have no singular points. By searching we can easily find three lines on the surface, defined by the following equations:

$$L_1 : x_0 = 9x_1 + x_4 = x_3 = x_5 = 0,$$
$$L_2 : x_0 = 9x_1 - x_4 = x_3 = x_5 = 0,$$
$$L_3 : 27x_0 + 2x_5 = 27x_1 - 3x_4 + 8x_5 = 3x_2 + x_4 - 2x_5 = 9x_3 + x_5 = 0,$$

(and with only slightly more difficulty we could find three more). Here L_2 meets L_1, L_3, but L_1 and L_3 are disjoint, so no elliptic fibration has a fibre supported on $L_1 \cup L_2 \cup L_3$. Letting H be the hyperplane class, we easily compute that $(H - [L_1] - [L_2] - [L_3])^2 = 0$, where $[L_i]$ is the Picard class of L_i, and as this class is easily checked to be free of base components we obtain an elliptic fibration from the linear forms $27x_0 + 2x_5, 9x_3 + x_5$ vanishing on all three lines. The reducible fibres

are four of type I_2, two of which consist of curves of degree $1, 4$ and two $2, 3$. Let C_1 be the conic in a reducible fibre that contains the point $(-4 : 1 : 0 : 0 : 9 : 0)$.

We are then fortunate to find that C_1, L_1, L_2 all meet transversely at $(0 : 0 : 1 : 0 : 0 : 0)$, thus forming a fibre of type IV. As in [2, Section 16], the associated fibration turns out in addition to have a rational I_2 fibre and six of type I_3, of which two are rational and the other four are conjugate over a quadratic extension of $\mathbb{Q}(\sqrt{37})$. There are two possible ways to match rational I_3 fibres on the two, only one of which is correct. As in the previous section, we find that the Jacobians of the general fibres are isomorphic. Since L_3 is a section, this is indeed an elliptic fibration, and so in fact the two fibrations are isomorphic as curves of genus 1 over $\mathbb{Q}(t)$. The defining polynomials are

$$(-x_0 - 2x_1 - x_2 + 2_3 : x_0 + 2x_1 + x_2 - x_3).$$

This completes the proof that the models of [7] are birationally equivalent to those of [2].

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